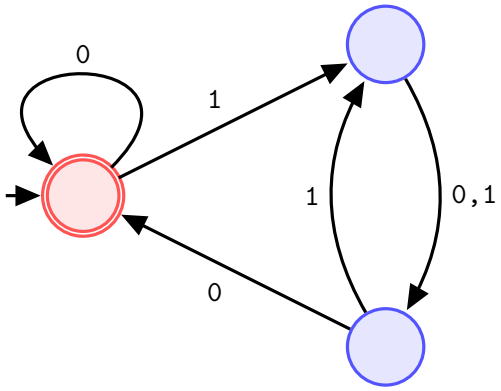
Basic

Recall the following language definitions from the lecture:

1. Acceptance problems
   1) $A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$
   2) $A_{\text{NFA}} = \{\langle B, w\rangle \mid B \text{ is an NFA that accepts input string } w\}$
   3) $A_{\text{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}$
   4) $A_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$
2. Language emptiness problems
   1) $E_{\text{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$
   2) $E_{\text{CFG}} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$
   3) $E_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$
3. Language equality problems
   1) $EQ_{\text{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$
   2) $EQ_{\text{CFG}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$
   3) $EQ_{\text{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

(1) This exercise is meant to help you get used to the *object encoding* notation.

Answer all parts for the following DFA M and give reasons for your answers.



1) Is $\langle M, 0100\rangle \in A_{\text{DFA}}$?
2) Is $\langle M, 011\rangle \in A_{\text{DFA}}$?
3) Is $\langle M\rangle \in A_{\text{DFA}}$?
4) Is $\langle M\rangle \in E_{\text{DFA}}$?
5) Is $\langle M, M\rangle \in EQ_{\text{DFA}}$?

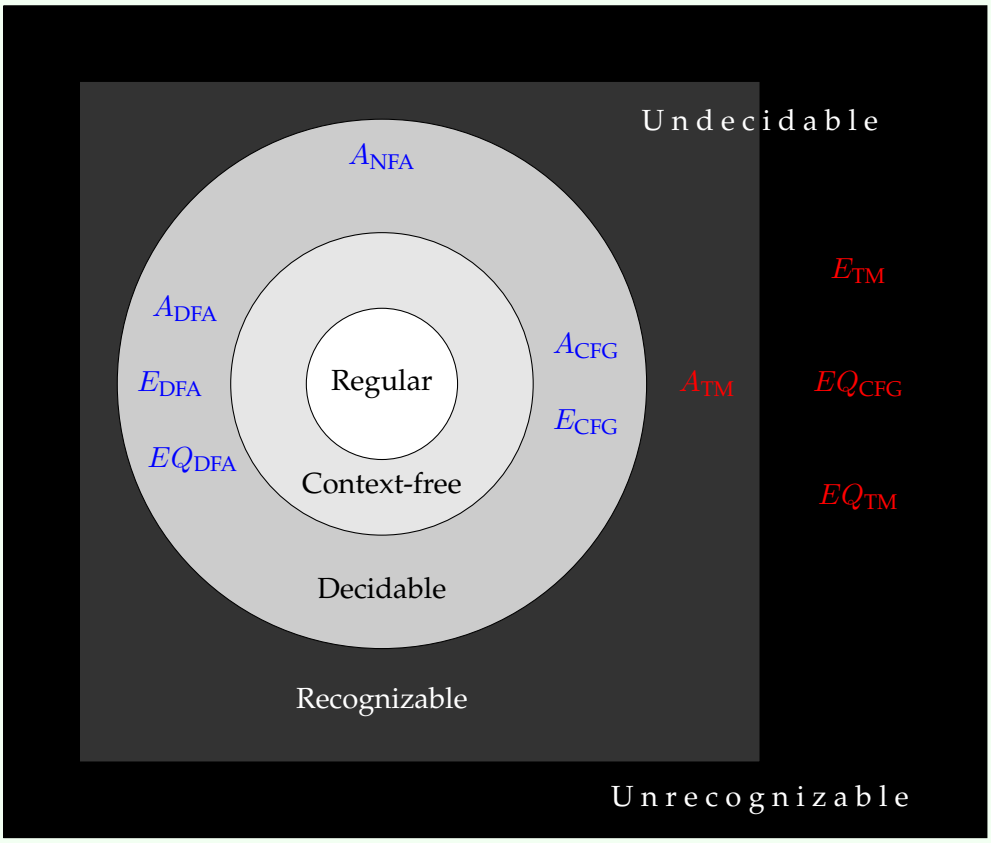
Solution

1) Is $\langle M, 0100\rangle \in A_{\text{DFA}}$? Yes. 0100 is accepted by M.
2) Is $\langle M, 011\rangle \in A_{\text{DFA}}$? No. 011 is **not** accepted by M.
3) Is $\langle M\rangle \in A_{\text{DFA}}$? No. Invalid encoding.
4) Is $\langle M\rangle \in E_{\text{DFA}}$? No. $L(M) = \{\varepsilon, 110, 100, 0100, \ldots\} \neq \emptyset$.
5) Is $\langle M, M\rangle \in EQ_{\text{DFA}}$? Yes. M and itself accept the same language.

(2) Draw a Venn diagram to illustrate the relationship between the languages: *regular, context-free, decidable*, and *recognizable*; then indicate where the following problems belong to:

$$A_{\text{DFA}}, E_{\text{DFA}}, EQ_{\text{DFA}}, \quad A_{\text{NFA}}$$
$$A_{\text{CFG}}, E_{\text{CFG}}, EQ_{\text{CFG}}$$
$$A_{\text{TM}}, E_{\text{TM}}, EQ_{\text{TM}}$$

Solution



(Whatever applies to DFAs also applies to NFAs because they are equivalent.)

(3) Read about the *Post Correspondence Problem (PCP)* on Wikipedia: http://en.wikipedia.org/wiki/Post_correspondence_problem.

- Is it decidable?
- How about when the alphabet is simply {a}?
- Find some (easy) examples and try to solve them by hand, e.g.

$$\left\{\left[\frac{\texttt{ab}}{\texttt{abab}}\right], \left[\frac{\texttt{b}}{\texttt{a}}\right], \left[\frac{\texttt{aba}}{\texttt{b}}\right], \left[\frac{\texttt{aa}}{\texttt{a}}\right]\right\}$$

- Try to write code to search for solutions using exhaustive search.

Solution

PCP is undecidable.

The special case where the alphabet consists of only one symbol is decidable.

The idea is as follows:

1) All tiles that have equal top and bottom strings can be ignored.
2) If all remaining tiles have top strings longer than bottom strings then the answer is False.
3) If all remaining tiles have top strings shorter than bottom strings then the answer is False.
4) If less than two dominoes are left then we have a trivial case to check.
5) Pick two dominoes $[\frac{x}{y}]$ and $[\frac{v}{w}]$ such that x is longer than y, and v is shorter than w.

Here is a sample Python script that could be used to try and find a solution to a given PCP instance. It works by trying all the possible combinations of length at most 20 tiles. Of course there may be a solution which requires longer combinations, or maybe the answer is False – code cannot rule out this case.

```
from itertools import product

tiles = [
    ("ab","abab"),
    ("b","a"),
    ("aba","b"),
    ("aa","a"),
]

def show_tile(tiles):
    ''' Pretty print the tiles combination '''
    top = bottom = middle = ''
    for tile in tiles:
        width = max( len(tile[0]),len(tile[1]) )
        top    += '| ' + tile[0].center(width) + ' '
        bottom += '| ' + tile[1].center(width) + ' '
        middle += '|..' + ('.'*width)
    print(top+'|')
    print(middle+'|')
    print(bottom+'|')
    print('\n')
```

```
print("The given tiles are (indexed as: {}):".format(
   list(range(len(tiles)))))
show_tile(tiles)

# Search all possible combinations up to length = 20
print("Solutions found so far:\n")
for length in range(1,21):
    for combination in product(tiles, repeat=length):
        top = bottom = ''
        for tile in combination:
            top    += tile[0]
            bottom += tile[1]
        if top == bottom:
            print(len(combination),'tiles:',[tiles.
   index(t) for t in combination],' -> ',top)
            show_tile(combination)
```

Here are some solutions found by the above script:

$$\left[\frac{\texttt{aa}}{\texttt{a}}\right]\left[\frac{\texttt{aa}}{\texttt{a}}\right]\left[\frac{\texttt{b}}{\texttt{a}}\right]\left[\frac{\texttt{ab}}{\texttt{abab}}\right] \rightarrow \texttt{aaaabab}$$

$$\left[\frac{\texttt{ab}}{\texttt{abab}}\right]\left[\frac{\texttt{ab}}{\texttt{abab}}\right]\left[\frac{\texttt{aba}}{\texttt{b}}\right]\left[\frac{\texttt{b}}{\texttt{a}}\right]\left[\frac{\texttt{b}}{\texttt{a}}\right]\left[\frac{\texttt{aa}}{\texttt{a}}\right]\left[\frac{\texttt{aa}}{\texttt{a}}\right] \rightarrow \texttt{abababbbaaaa}$$

$$\left[\frac{\texttt{aa}}{\texttt{a}}\right]\left[\frac{\texttt{aa}}{\texttt{a}}\right]\left[\frac{\texttt{b}}{\texttt{a}}\right]\left[\frac{\texttt{ab}}{\texttt{abab}}\right]\left[\frac{\texttt{aa}}{\texttt{a}}\right]\left[\frac{\texttt{aa}}{\texttt{a}}\right]\left[\frac{\texttt{b}}{\texttt{a}}\right]\left[\frac{\texttt{ab}}{\texttt{abab}}\right] \rightarrow \texttt{aaaababaaaabab}$$

(4) Let $AMBIGCFG = \{\langle G\rangle \mid G \text{ is an ambiguous CFG}\}$.

Show that *AMBIGCFG* is undecidable.

Hint: Use a reduction from PCP (above) as follows:

Given an instance

$$\left\{\left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \cdots, \left[\frac{t_k}{b_k}\right]\right\}$$

of the PCP, construct a CFG G with the rules:

$$\begin{array}{lcl} S & \rightarrow & T \mid B \\ T & \rightarrow & t_1 T a_1 \mid \cdots \mid t_k T a_k \mid \varepsilon \\ B & \rightarrow & b_1 B a_1 \mid \cdots \mid b_k B a_k \mid \varepsilon \end{array}$$

where $a_1, a_2, \ldots, a_k$ are new terminal symbols.

Solution

Let us *reduce* PCP to *AMBIGCFG*.

Let:

$$\left\{\left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \cdots, \left[\frac{t_k}{b_k}\right]\right\}$$

be a given instance of PCP.

Let us first consider the following grammar G designed to mimic the juxtapo-

sition of the tiles (T for the top patterns, and B for the bottom patterns):

$$\begin{array}{lcl} S & \rightarrow & T \mid B \\ T & \rightarrow & t_1 T a_1 \mid \cdots \mid t_k T a_k \mid \varepsilon \\ B & \rightarrow & b_1 B a_1 \mid \cdots \mid b_k B a_k \mid \varepsilon \end{array}$$

where $a_1, a_2, \ldots, a_k$ are new terminal symbols used to keep track of which tiles were used.

When a string is generated it will be of the form $t_i t_j t_k \cdots a_k a_j a_i$ (if we start with $S \rightarrow T$) or $b_i b_j b_k \cdots a_k a_j a_i$ (if we start with $S \rightarrow B$). If these two strings are equal then we know they have been generated in two different ways, and so this grammar must be ambiguous.

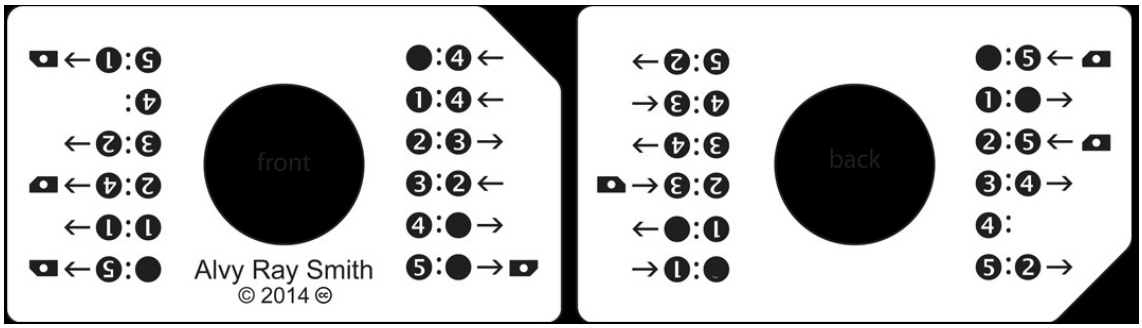
There is, however, a slight problem with this grammar as it allows the generation of ε which corresponds to the empty solution of PCP. To fix this we change G as follows (to ensure that at least one tile is used):

$$\begin{array}{lcl} S & \rightarrow & T \mid B \\ T & \rightarrow & t_1 T a_1 \mid \cdots \mid t_k T a_k \mid t_1 a_1 \mid \cdots \mid t_k a_k \\ B & \rightarrow & b_1 B a_1 \mid \cdots \mid b_k B a_k \mid b_1 a_1 \mid \cdots \mid b_k a_k \end{array}$$

Now, if $AMIGCFG$ were decidable then we would map the given PCP instance to this new grammar G then use $AMIGCFG$'s decider to decide if G is an ambiguous or not. If the answer is True then the PCP instance is also solvable, otherwise it is not. But since PCP is not decidable then such a decider cannot exist.

We conclude that $AMIGCFG$ is not decidable.

(1) Here is a **Universal Turing Machines** (UTM) as business card TM!



This TM is... **universal** with only 4 states and 6 symbols!

This TM can simulate any other TM!

→ a **computer** is a simple idea – it is programming it that is hard...

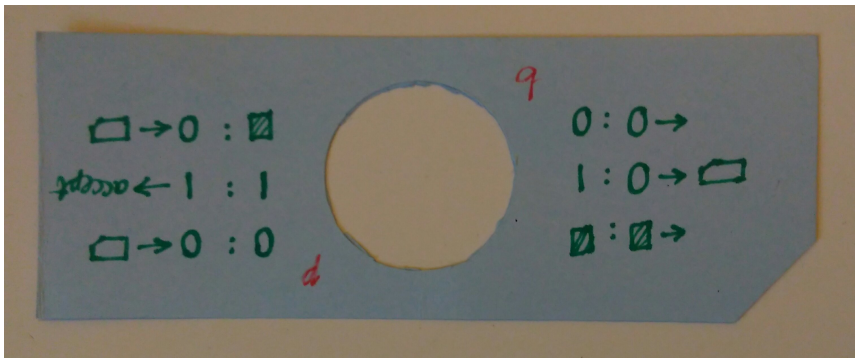
Explanation of how to use this machine can be found at: http://alvyray.com/CreativeCommons/TuringToys.htm

(2) **Business card TMs**

Recall the TM from last week over $\Sigma = \{0, 1\}$ with $Q = \{q, p, q_{\text{accept}}, q_{\text{reject}}\}$, $q_{\text{start}} = q$, $\Gamma = \{0, 1, \square\}$, and δ:

State	Tape symbol	Transition
q	0	$(q, 0, R)$
q	1	$(p, 0, R)$
q	$\square$	$(q, \square, R)$
p	0	$(q, 0, L)$
p	1	$(q_{\text{accept}}, 1, R)$
p	$\square$	$(q, 0, L)$

Use cardboard to make a "business card" representation of it as follows:



To use it, you would place it on the memory tape with the current cell visible through the circular opening and the current state showing on the right, etc. as in the previous question.

(3) "A *quine* is a computer program which takes no input and produces a copy of its own source code as its only output. The standard terms for these programs in the computability theory and computer science literature are *self-replicating programs*, *self-reproducing programs*, and *self-copying programs*." http://en.wikipedia.org/wiki/Quine_(computing)

Write a quine in your preferred programming language.

This is related to the concept of *re____ion*. (Find the missing letters.)