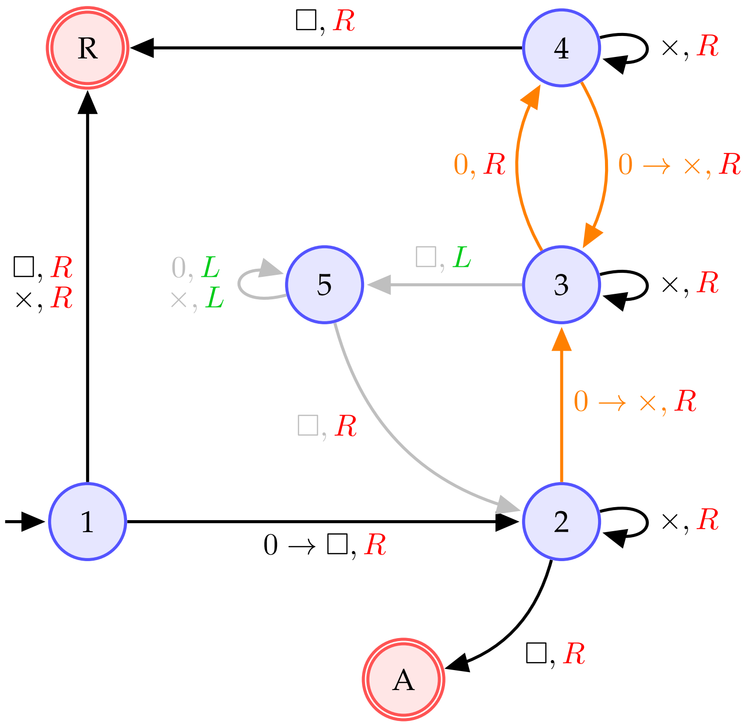
(1) Recall the language $\{0^{2^n} \mid n \geq 0\} = \{0, 00, 0000, 00000000, 0^{16}, 0^{32}, 0^{64}, \ldots\}$ from the lecture. The language L consisting of all strings of 0's whose length is a power of 2.

The formal description of the corresponding TM is:

- $Q = \{1, 2, 3, 4, 5, \mathrm{A}, \mathrm{R}\}$
- $\Sigma = \{0\}$
- $\Gamma = \{0, \times, \square\}$
- The start, accept and reject states are 1, A and R, respectively.

$$\begin{aligned} q_{\text{start}} &= 1 \\ q_{\text{accept}} &= A \\ q_{\text{reject}} &= R \end{aligned}$$

- δ is given by the state diagram:



Notation:
$\mathtt{a} \to \mathtt{b}, R$:
read $\mathtt{a}$, write $\mathtt{b}$,
R: move right.
L: move left.

$\mathtt{a}, R$: $\mathtt{a} \to \mathtt{a}, R$

Trace this TM on the following inputs:

$$0, 0^2, 0^3, 0^4, 0^5, 0^6, 0^7, 0^8, 0^9, 0^{10}, 0^{11}, 0^{12}$$

(For each string, write the sequence of "configurations" taken by the TM.)

Solution

For 0 we get the following sequence of *configurations*:

$$10\square \to \square 2\square \to \square\square A$$

For 00 we get:

$$100\square \to \square 20\square \to \square\times 3\square \to \square 5\times\square \to 5\square\times\square \to \square 2\times\square \to \square\times 2\square \to \square\times\square \mathtt{A}$$

For the rest use JFLAP or the associated Python script, as the sequences are longer.

(2) You are given a TM where:

- $Q = \{q, p, q_{\text{accept}}, q_{\text{reject}}\}$
- $q_{\text{start}} = q$
- $\Sigma = \{0, 1\}$
- $\Gamma = \{0, 1, \square\}$
- δ is given by the following table:

State	Tape symbol	Transition
q	0	$(q, 0, R)$
q	1	$(p, 0, R)$
q	$\square$	$(q, \square, R)$
p	0	$(q, 0, L)$
p	1	$(q_{\text{accept}}, 1, R)$
p	$\square$	$(q, 0, L)$

For example (second row in the table), if the TM is in state q and the currently read symbol is 1 then the TM changes its state to state p, writes 0 (replaces 1 with 0) and then moves to the right.

1) Draw the state diagram of this TM.
2) Describe the property that input strings must have for this TM to halt, i.e. go into the accept or reject states.
3) Identify a string that makes it halt from the list below.

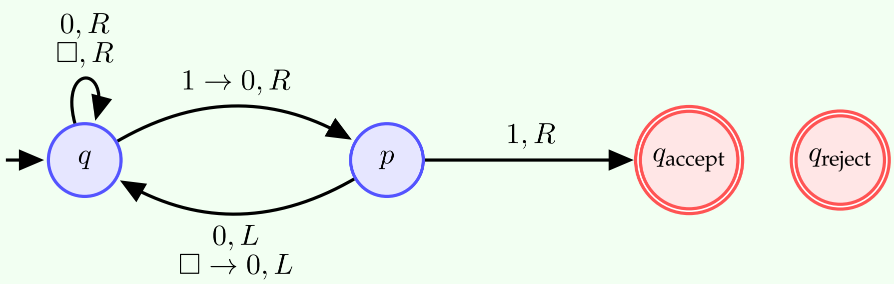
$$0000 \qquad 0100 \qquad 1010 \qquad 0110$$

4) Simulate this TM on the input 1010110, and identify which one of the following configurations is valid.

$$00000000q\square \qquad 00000p10 \qquad 10q10110 \qquad 000000p0$$

Solution

1) State diagram



2) Accepts strings that contain 11.

   If a string is not accepted it *loops* (goes on and on without stopping)

   Rejects nothing!

3) 0110
4) $00000p10$

(3) A non-deterministic TM with start state q_0 has the following transition function:

	0	1	$\square$
q_0	$\{(q_1, 0, R)\}$	$\{(q_1, 0, R)\}$	$\{(q_1, 0, R)\}$
q_1	$\{(q_1, 1, R), (q_2, 0, L)\}$	$\{(q_1, 1, R), (q_2, 1, L)\}$	$\{(q_1, 1, R), (q_2, \square, L)\}$
q_2	$\{(q_{\text{accept}}, 0, R)\}$	$\{(q_2, 1, L)\}$	$\{(q_{\text{reject}}, \square, R)\}$

1) Draw the state diagram of this TM.

2) Simulate all sequences of 5 moves, starting from initial configuration $q_0 1010$.

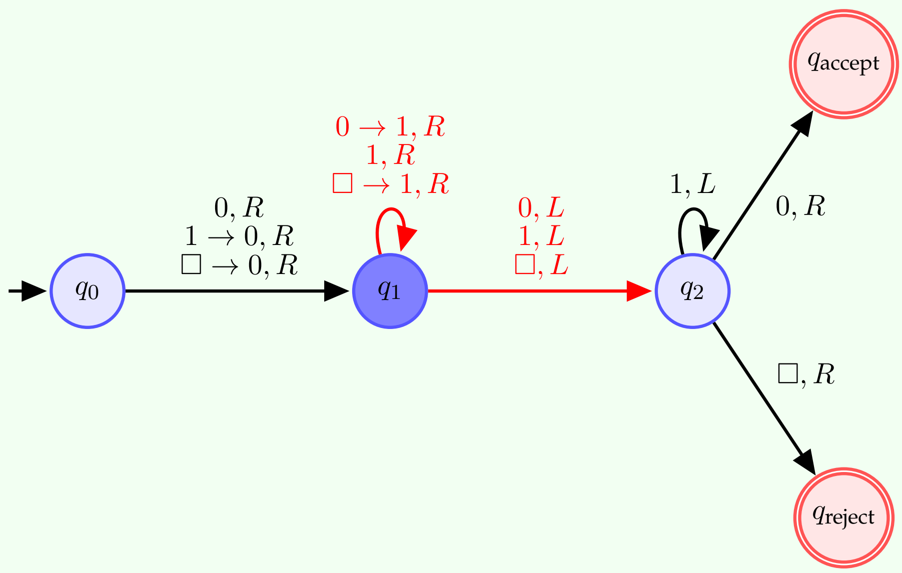
(NB. `JFLAP` does not handle non-deterministic TMs.)

3) Find, in the list below, one of the configurations reachable from the initial configuration in **exactly** 5 moves.

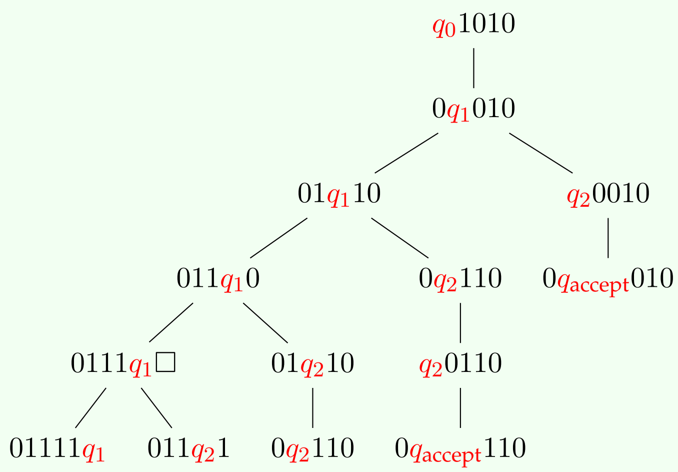
$q_2 0110$ $\quad$ $0 q_{\text{accept}} 110$ $\quad$ $011111 q_1$ $\quad$ $0111 q_2 1$

Solution

1) State diagram



2)



3) $0 q_{\text{accept}} 110$

Optional

(1) (**The Busy Beaver problem**) This is a very interesting and fun problem! Start by watching the following videos:

- https://www.youtube.com/watch?v=DILF8usqp7M
- https://www.youtube.com/watch?v=CE8UhcyJS0I
- https://www.youtube.com/watch?v=ZiTeuZSDB0U

You may also want to have a look at https://arxiv.org/abs/0906.3749 (The Busy Beaver Competition: a historical survey).

Can you produce the first few busy beavers? Compete with your friends!

(2) Play with the TM simulator at http://turingmaschine.klickagent.ch

First, observe and try to understand how the multi-tape TMs work, then how the same operations are done on one tape.

Can you see how to design a TM that on input 1^n produces 1^{n^2} using 2 or 1 tape(s)?

Solution

For 1^{n^2} we run the multiplication TM to compute $n \times n$ (TM would run on on 1^n and 1^n).