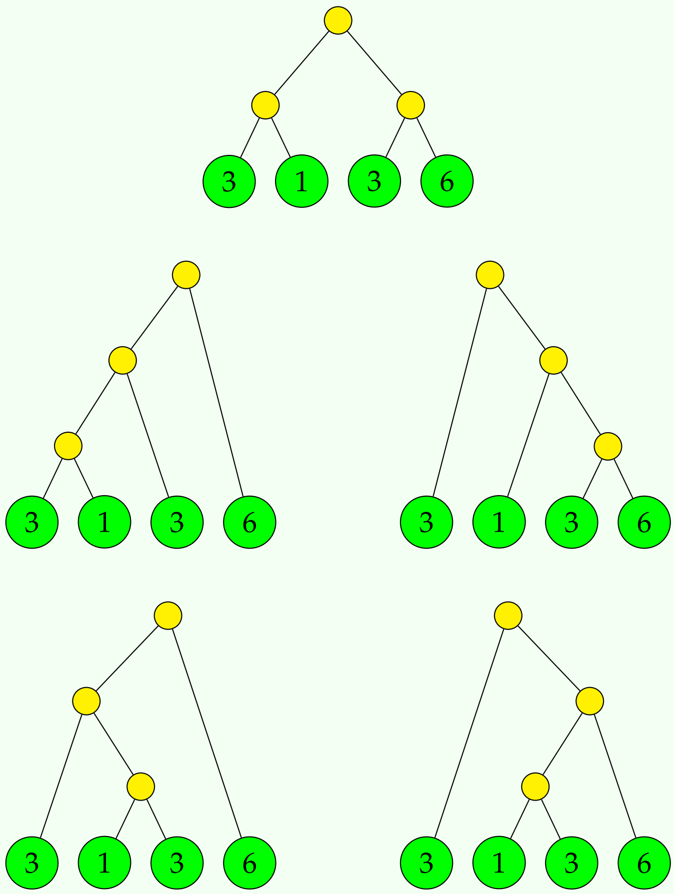
(1) Add the missing arithmetic operators $(+, -, \times, /)$ and parentheses to the following expression to make it true:

$$3 \quad 1 \quad 3 \quad 6 \quad = \quad 8.$$

Solution

$$(3+1)/3\times 6 \quad = \quad (3+1)/(3/6) \quad = \quad 8$$

We can exhaustively search all the possibilities by considering all the possible "arithmetic trees" on these 4 terms/factors:



where the yellow vertices can be any of the four arithmetic operations $(+ - \times /)$.

Points to note:

- Not easy to find, but easy to verify. ("NP certificates")
- Think outside the box: think about fractions, not only integers.
- Could do exhaustive search (Arithmetic trees), but may be easier to "guess then check."

(2) A little girl counts from 1 to 1000 using the fingers of her left hand as follows. She starts by calling her thumb 1, the first finger 2, middle finger 3, ring finger 4, and little finger 5. Then she reverses direction, calling the ring finger 6, middle finger 7, the first finger 8, and her thumb 9, after which she calls her first finger 10, and so on. If she continues to count in this manner, on which finger will she stop?

Solution

Let us simulate the first few numbers on a spreadsheet:

Thumb	First finger	Middle finger	Ring finger	Little finger
1				
	2			
		3		
			4	
				5
			6	
		7		
	8			
9				
	10			
		11		
			12	
				13
			14	
		15		
	16			
17				
	18			
		19		
			20	
				21
			22	

Aha! The numbers are distributed as follows:

Thumb:	$1 + 8k$ for $k \geq 0$.
First Finger:	$2 + 8k$ and $8k$
Middle finger:	$3 + 8k$ and $7 + 8k$
Ring finger:	$4 + 8k$ and $6 + 8k$
Little finger:	$5 + 8k$

So looking at the numbers "modulo 8" will tell us on which finger it will fall. ("modulo" means remainder after division).

For 1000 we have $1000 = 125 \times 8$, so it will fall on the first finger.

Your ability to analyse problems like this will help you analyse algorithms and estimate their running time.

(3) There are 100 closed lockers in a hallway. A man begins by opening all one hundred lockers. Next, he closes every second locker. Then he goes to every third locker and closes it if it is open or opens it if it is closed.

He continues like this until his 100^{th} pass in the hallway, in which he only changes the state of locker number 100.

How many lockers will be left open at the end?

Solution

Here is a simulation of the first parts of this:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0		All closed initially.
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1		A man begins by opening all the lockers.
3	1	0	1	0	1	0	1	0	1	0	1	0	1	0		He closes every second locker.
4	1	0	0	0	1	1	1	0	0	0	1	1	1	0		He flips the state of every 3rd locker.
5	1	0	0	1	1	1	1	1	0	0	1	0	1	0		He flips the state of every 4th locker.
6	1	0	0	1	0	1	1	1	0	1	1	0	1	0		He flips the state of every 5th locker.
7	1	0	0	1	0	0	1	1	0	1	1	1	1	0		He flips the state of every 6th locker.
8	1	0	0	1	0	0	0	1	0	1	1	1	1	1		He flips the state of every 7th locker.
9	1	0	0	1	0	0	0	0	0	1	1	1	1	1		He flips the state of every 8th locker.
10	1	0	0	1	0	0	0	0	1	1	1	1	1	1		He flips the state of every 9th locker.
11	1	0	0	1	0	0	0	0	1	0	1	1	1	1		He flips the state of every 10th locker.
12	1	0	0	1	0	0	0	0	1	0	0	1	1	1		He flips the state of every 11th locker.
13	1	0	0	1	0	0	0	0	1	0	0	0	1	1		He flips the state of every 12th locker.
14	1	0	0	1	0	0	0	0	1	0	0	0	0	1		He flips the state of every 13th locker.
15	1	0	0	1	0	0	0	0	1	0	0	0	0	0		He flips the state of every 14th locker.
16																
17			0	=	closed											
18			1	=	open											
19				=	flipped state (closed -> open, or: open -> closed)											

We can see from this that lockers number 1, 4 and 9 remain open.

Let us now write a quick Python script to simulate the operations:

```
lockers = [False for i in range(101)]

for run in range(1,101):
    for locker in range(run,101,run):
        lockers[ locker ] = not lockers[
            locker ]

for i in range(101):
    if lockers[i]:
        print(i)
```

The lockers that remain open are: $1, 4, 9, 16, 25, 36, 49, 64, 81, 100$.

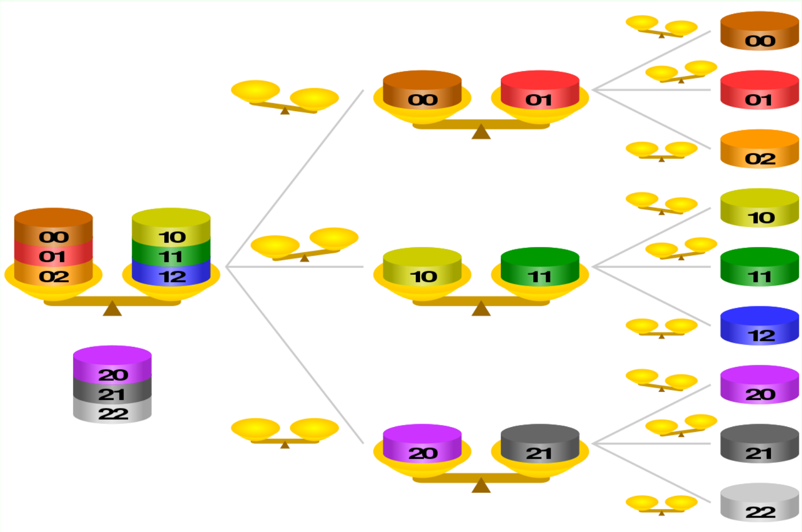
10 lockers: $1^2, 2^2, 3^3, \ldots, 10^2$ because locker ℓ remains open iff it has an odd number of divisors, which only happens if ℓ is a perfect square.

- Quick code helps find the pattern. Use Maths to justify.

(4) There are eight identical-looking coins; one of these coins is counterfeit and is known to be lighter than the genuine coins. What is the minimum number of weighings needed to identify the fake coin with a two-pan balance scale without weights?

Solution

It can be done in 2 steps. The following diagram illustrates the method for 9 coins, but the same applies to 8 coins (just ignore coin 22 for example).



Select from the given coins two groups of three coins each and put them on the opposite cups of the scale. If they weigh the same, the fake is among the other two coins, and weighing these two coins will identify the lighter fake.

If the first weighing does not yield a balance, the lighter fake is among the three lighter coins. Take any two of them and put them on the opposite cups of the scale. If they weigh the same, it is the third coin in the lighter group that is fake; if they do not weigh the same, the lighter one is the fake.

Since the problem cannot be solved in one weighing, the above algorithm requiring just two weighings is optimal.

- Halving the problem size usually yields efficient algorithms (binary search), but this is a rare example dividing the problem into thirds is better. So don't be fooled!
- Algorithmic Puzzles gives an alternative solution in which the second weighing does not depend on the results of the first one:

  Label the coins by the letters A, B, C, D, E, F, G, H.

  On the first weighing, weigh A, B, C against F, G, H.
  On the second weighing, weigh A, D, F against C, E, H.

  - If $ABC = FGH$ (the first weighing results in a balance), all these six coins are genuine, and therefore the second weighing is equivalent to weighing D against E.
  - If $ABC < FGH$, only A, B, C may still be fake. Therefore
    * if on the second weighing $ADF = CEH$, B is the fake;
    * if $ADF < CEH$, A is the fake;
    * and if $ADF > CEH$, C is the fake.
  - The case of $ABC > FGH$ is symmetric to the case just discussed.

- You have a 5 litre jug and a 3 litre jug, and an unlimited supply of water, but no measuring cups.

  How would you come up with exactly 4 litre of water?

Solution

Using a table form we can simulate the operations as follows:

Action	A (5L)	B (3L)
	0	0
Fill A	5	0
Empty A into B	2	3
Empty B	2	0
Empty A into B	0	2
Fill A	5	2
Empty A into B	4	3

There are many other (longer) solutions too.

  - Not easy to find, easy to verify. (“NP certificates”)
  - Could do exhaustive search (Turn into a graph), but may be easier to “guess then check.”

- Little Alice has 10 pockets and £44 in £1 coins.

  She wants to put her coins in her pockets so distributed that each pocket contains a different number of pounds. Can she do so?

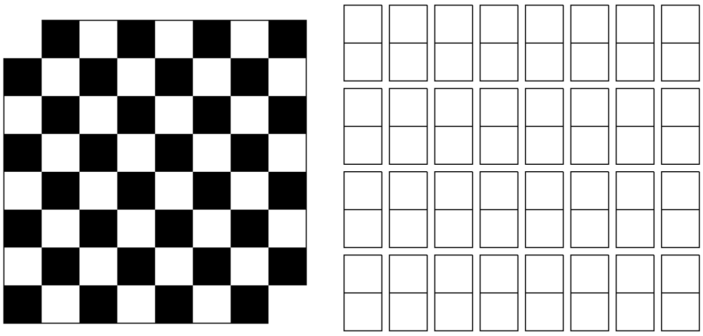
  Solution

  No.

  Minimum is:
  $$0 + 1 + 2 + \cdots + 9 = \frac{10 \times (0 + 9)}{2} = 45 > 44.$$

  - Sometimes we have to prove a solution does not exist.
  - This method is called "proof by contradiction." We assume something is possible/true and then derive a contradiction. We then conclude that the assumption must have been wrong.

- Below is an 8×8 chess board in which two diagonally opposite corners have been cut off.



You are given plenty of dominoes, such that each domino can cover exactly two squares.

Can you cover the entire board with dominoes? (No dominoes are allowed to overlap or be partly outside the board.)

Can you *prove* your answer is correct? (Show an example solution if this is possible, or show that it is impossible.)

Solution

No.

There are more black squares than white squares. (Each domino covers exactly one black and one white).

– Idea of *invariant*.