Limitations of the Regular Languages

The Pumping Lemma

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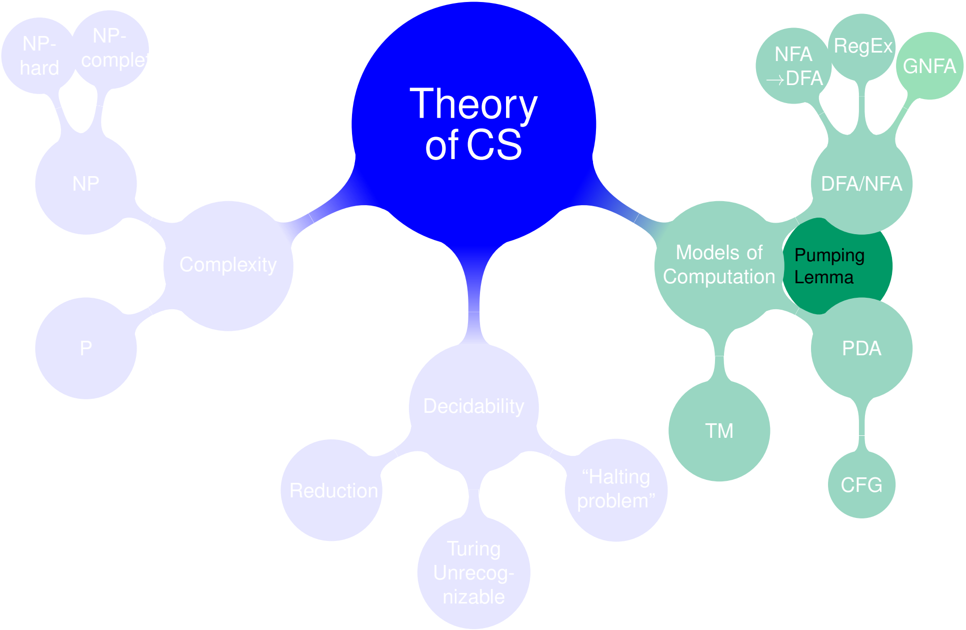
Lecture 4


NP-hard
NP-complete
NP
Complexity
P
Theory of CS
Decidability
Reduction
"Halting problem"
Turing Unrecognizable
NFA →DFA
RegEx
GNFA
DFA/NFA
Models of Computation
Pumping Lemma
PDA
TM
CFG

Last week. . .

Regular Languages

The class of regular languages can be:

1. Recognized by NFAs. (equiv. GNFA or ε-NFA or NFA or DFA).
2. Described using **Regular Expressions**.

Today:

1. See the limit of regular languages.
2. How to show a language is not regular.

Types of proofs

We show a language is regular using "**proof by existence**":

- Construct an NFA recognizing it.
- Write a Regular Expression for it.

  Using closure under the **union**, **concatenation** and **star** operations.

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However, if a languages is *not regular* then how can we show that?!

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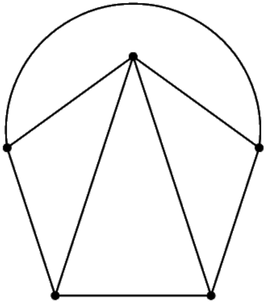
- Thus, it is **not** raining!

Eulerian paths – example of proof by contradiction

Is it possible to traverse this graph by travelling along **each edge exactly once**?



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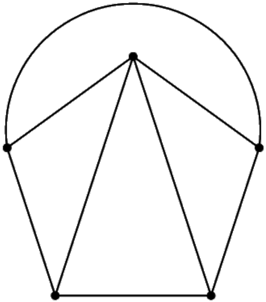

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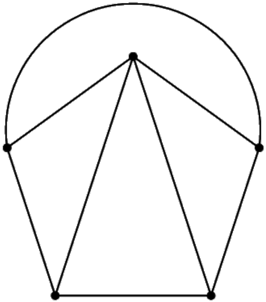

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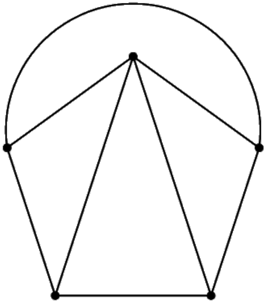

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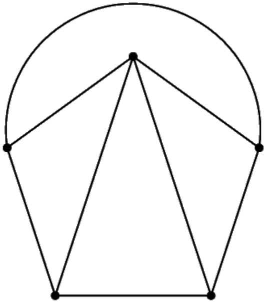

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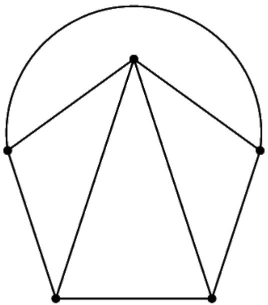

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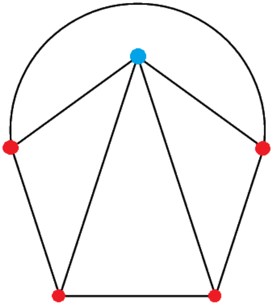

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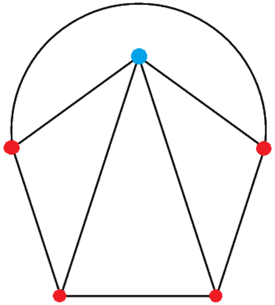

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Pumping Lemma

Mindmap
Proofs
Proof by existence
Proof by contradiction
Observation
Unary alphabet
Pigeon-hole principle
Binary alphabet
Pumping Lemma
PL Game!
Examples
$a^n b^n$
ww
Pumping down
Implications
Constant Space

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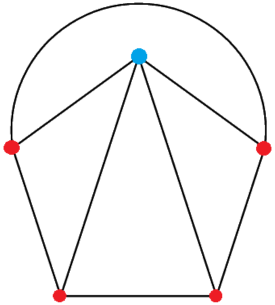

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  - There can only be 0 or 2 such exceptions.
- However, this graph has 4 exceptions!
- Thus, it is impossible to traverse this graph by travelling along each path exactly once.

Types of proofs – Proof by contradiction

To prove a language is not regular, we can use **proof by contradiction**.

- We need a **property** that all regular languages must satisfy.
- Then, if a given language does not satisfy it then it cannot be regular.

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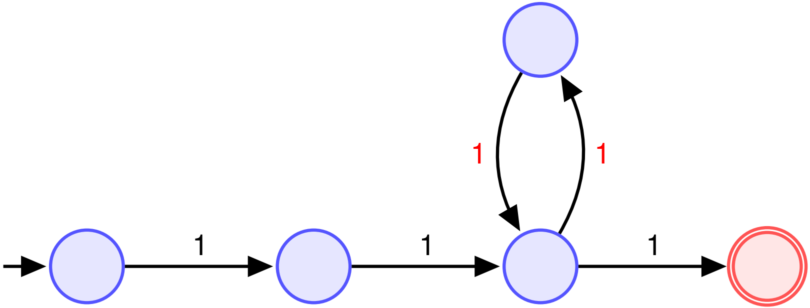
Let us try to understand the regular languages (RLs) a bit more. . .

- Let us examine some examples in the next few slides. . .
- For each automaton, let us think about the **path taken by an accepted string** – is it "straight" or does it loop?

Unary alphabet $\{1\}$ – Strings of length $3, 5, 7, 9, \ldots$



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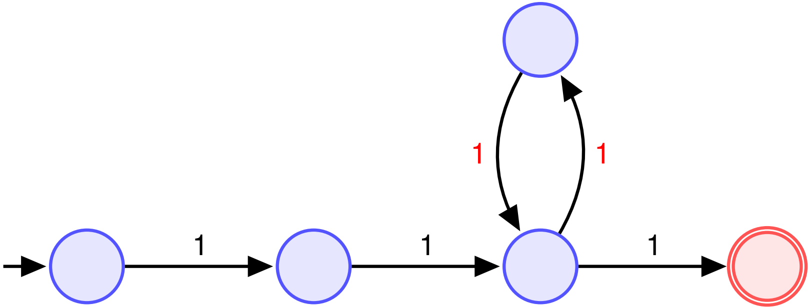

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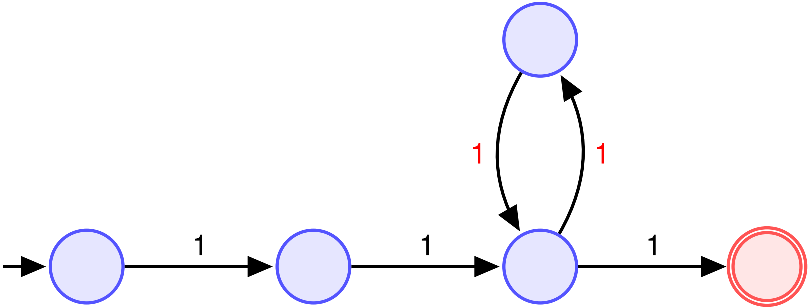

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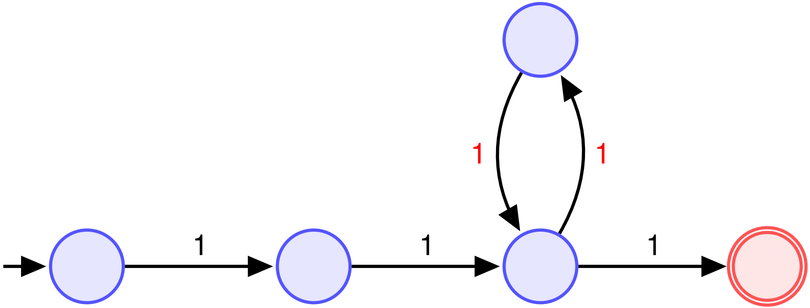

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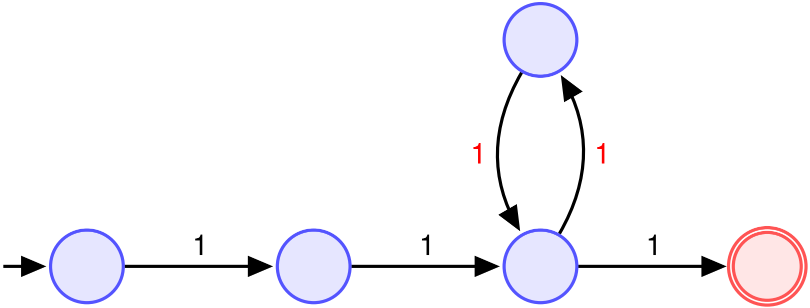

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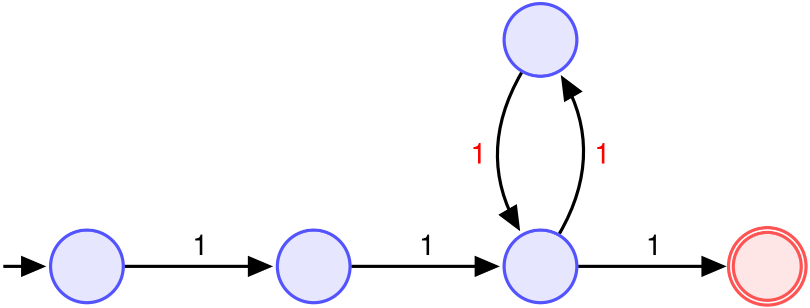

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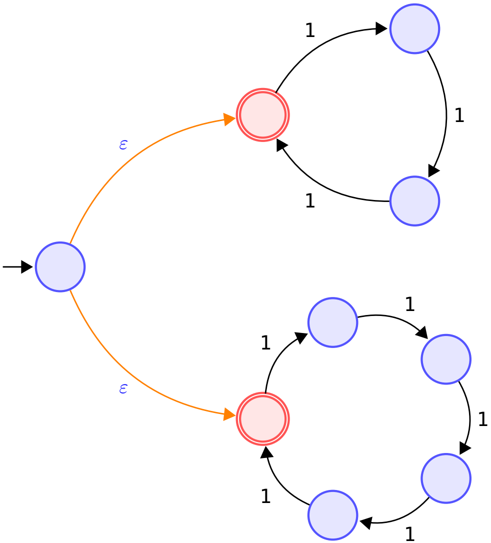
We say: we **pump** the substring 11.

$(111)^* + (11111)^*$



Set of accepted strings is:
$\{\varepsilon, 1^3, 1^6, 1^9, \ldots\} \cup \{\varepsilon, 1^5, 1^{10}, 1^{15}, \ldots\}$

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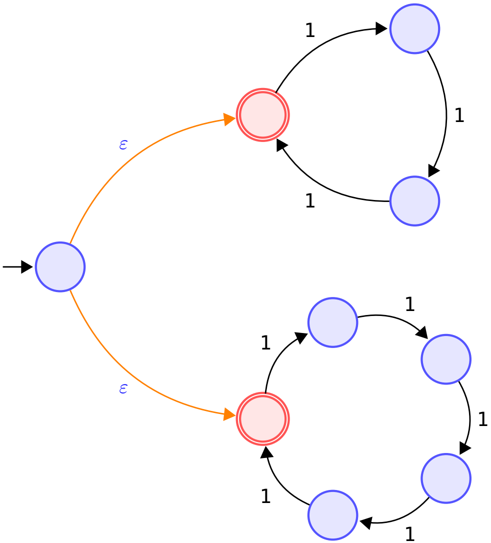

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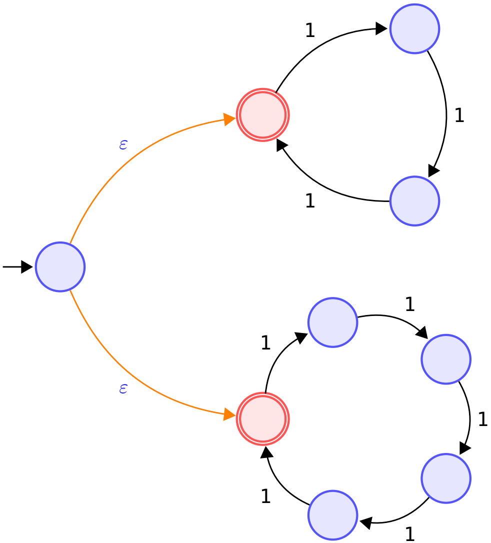

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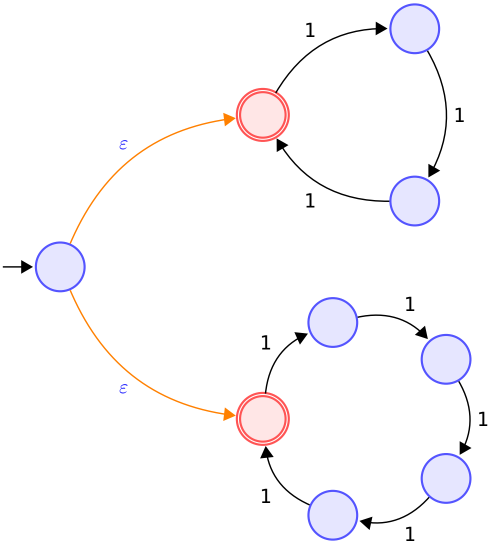

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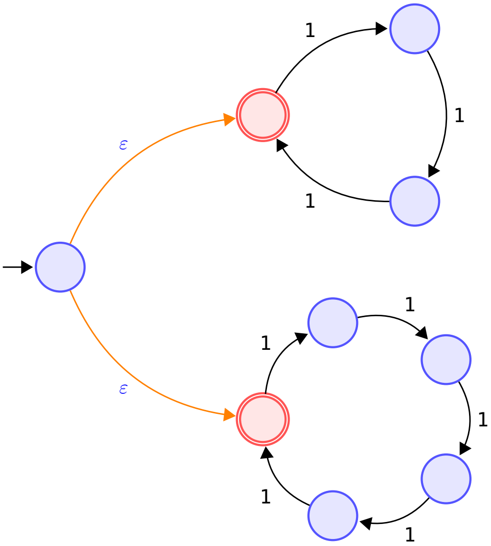

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- 3, the length of 111, is called: **pumping length.**

Unary alphabet

Let L be a regular language over a unary alphabet $\Sigma = \{1\}$.
The language L is:

- either **finite**, in which case it is regular trivially,
- or **infinite**, in which case its DFA will have to **loop**:
  - The DFA that recgnizes L has a finite number of states.
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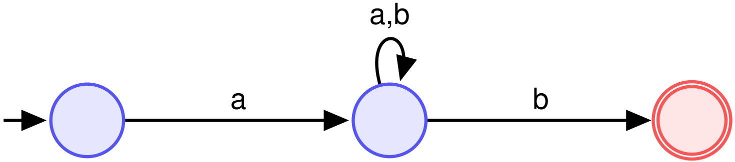
Pigeon-hole principle

If we put **more than** n pigeons into n holes then there must be a hole with more than one pigeon in.

$a\Sigma^*b$



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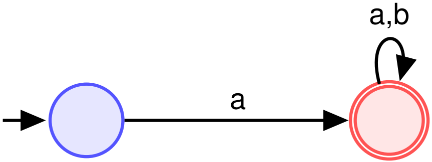

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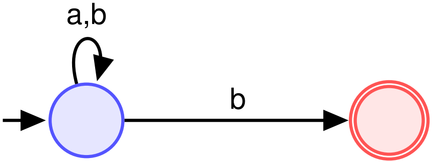
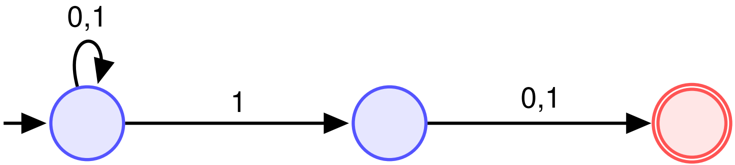
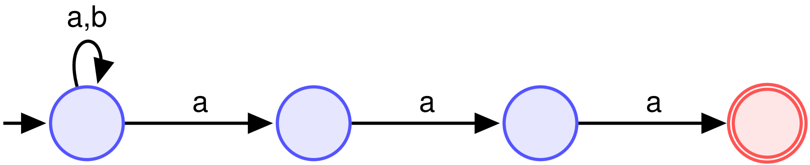


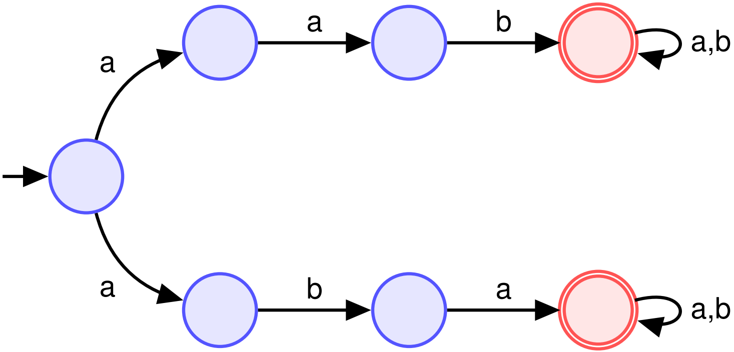
$\Sigma^* 1 \Sigma$



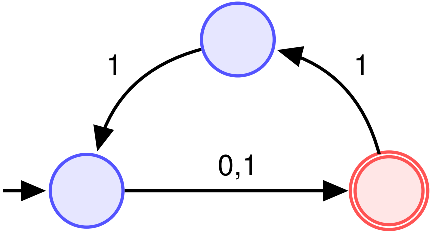
$\Sigma^* aaa$



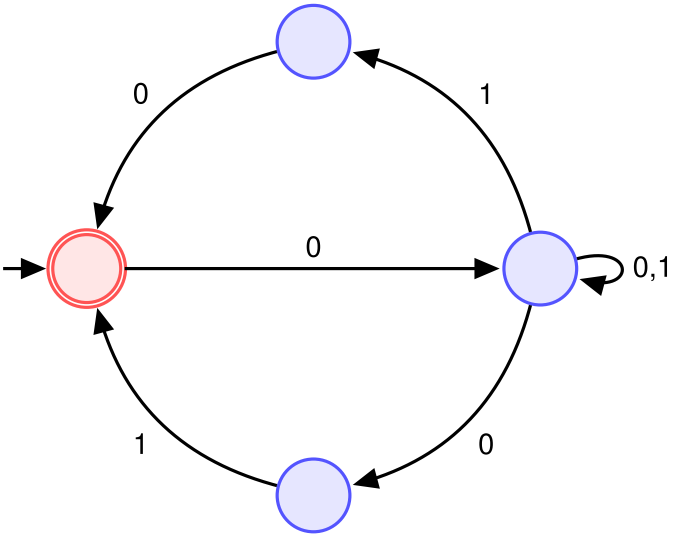
$aab\Sigma^* + aba\Sigma^*$



$(\Sigma 11)^*\Sigma$



$(0\Sigma^*(01+10))^*$



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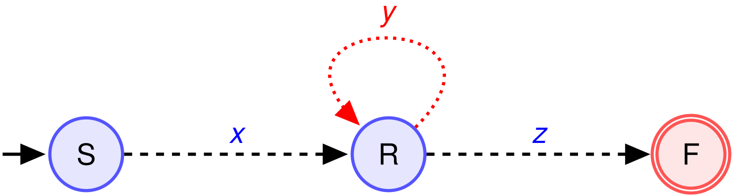

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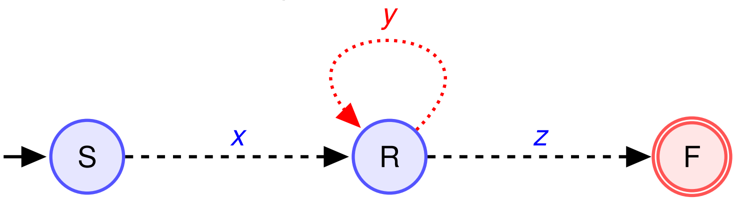

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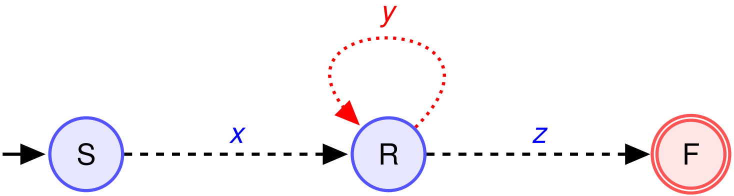

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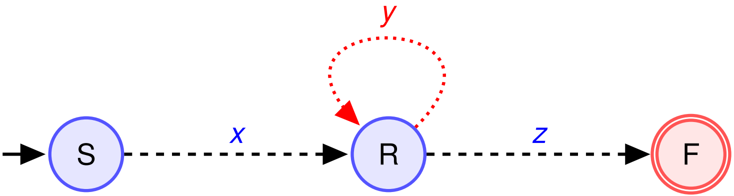

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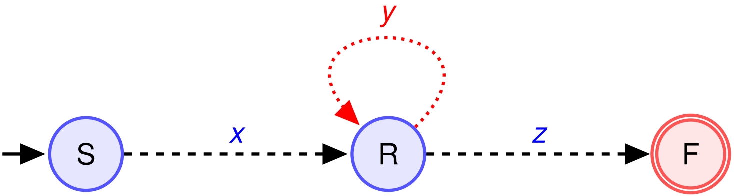

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For any RL L, it is possible to divide an **accepted string**, that is "long enough", into 3 substrings xyz, in such a way that xy^*z is a subset of L.

The Pumping Lemma (PL)

- We will denote a **pumping length** by p.
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Let L be a regular language. Then, there exists a constant p such that every string w from L, with $|w| \geq p$, can be broken into three substrings xyz such that

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Its main purpose in practice is to prove that a language is **not** regular.

That is, if we can show that a language L does not have this property, then we conclude that L cannot be recognized by a DFA/NFA or expressed as a regular expression.

The PL Game!

When the PL is used to prove that a language L is **not regular**, the proof can be viewed as a "game" between a **Prover** and a **Falsifier** as follows:

The PL Game!

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If **Falsifier** always wins then L is **not regular**.
If **Prover** always wins then L **may be** regular.

Example ($L = \{a^n b^n \mid n \geq 0\}$)

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$$w = \underbrace{a \ldots\ldots\ldots\ldots a}_{p \text{ symbols}}\underbrace{b \ldots\ldots\ldots b}_{p \text{ symbols}}$$

❸ **Prover** tries to split $w = a \ldots ab \ldots b$ into xyz such that $|xy| \leq p$

$$\underbrace{a \ldots}_{x}\underbrace{\ldots\ldots\ldots}_{y}\underbrace{\ldots ab \ldots\ldots b}_{z}$$

Since y must be within the first p symbols then y is made of a's only.

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❹ **Falsifier** now can for example build

$$xy^2z = xyyz = \underbrace{a \ldots\ldots\ldots\ldots a}_{\text{more than } p \text{ symbols}}\underbrace{b \ldots\ldots\ldots b}_{\text{still } p \text{ symbols}}$$

Hence $xy^2z \notin L$, and L is not regular.

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$$w = \underbrace{0 \ldots\ldots\ldots\ldots 0}_{p \text{ symbols}} 1 \underbrace{0 \ldots\ldots\ldots 0}_{p \text{ symbols}} 1$$

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❹ **Falsifier** pumps y to produce

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❹ **Falsifier** pumps y **down** and forms $xy^0z = xz$

$$xy^0z = \underbrace{a\ldots\ldots\ldots a}_{\text{at most } p \text{ symbols}}\underbrace{b\ldots\ldots\ldots b}_{\text{still } p \text{ symbols}}$$

Hence $xy^0z \notin L$, and L is not regular.

Food for thought

If “modern computer” = Finite Automaton then:

- We can only store a fixed finite amount of data, say $1\text{TB} = 1024^4 \times 8 = 2^{43}$ **bits** of information, i.e. a maximum of $2^{2^{43}} \approx 10^{2,647,887,844,335}$ **states** – a finite number still!

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- We have assumed that the input string is not stored in the computer. . . (otherwise, it would just run out of memory anyway).
- However, at 3GHz for example, this would take. . . a length of time so inconceivably huge that the age of the universe would be negligible by comparison. (So, do we care?)

Space Complexity: Constant Space $\longleftrightarrow$ NFAs

- Finite Automaton: good model for algorithms which require **constant space** (i.e. space used does not grow with respect to the input size).
- Some languages cannot be recognized by NFAs.

  *Space used in computation must **grow** with respect to the input size.*
- We will see a more powerful model of computation next week!