Models of Computation: DFAs & NFAs

Deterministic/Non-deterministic Finite Automata

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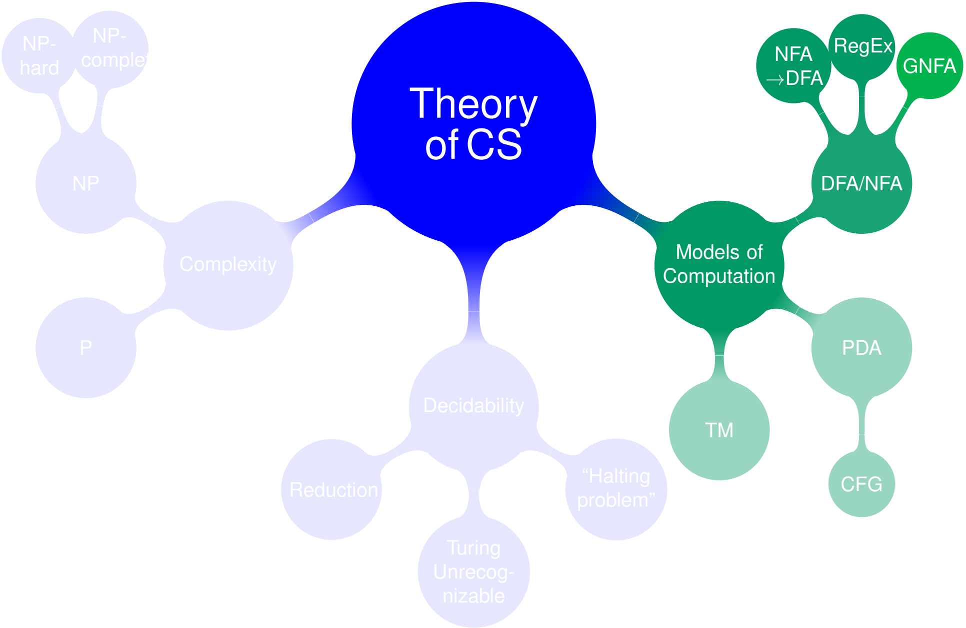
Lecture 2


NP-hard
NP-complete
NP
P
Complexity
Theory of CS
Decidability
Reduction
"Halting problem"
Turing Unrecog-nizable
Models of Computation
DFA/NFA
NFA →DFA
RegEx
GNFA
PDA
CFG
TM

“Problems”. . .

Last week: We can focus on **decision problems** only.

Decision problems

A yes/no question on a set of inputs.

Given a **search space** and a desired **property**,
decide whether the *search space* contains an item with that *property* or not.

Encoding problems

- The question will be presented as a **string**:

  A *sequence of symbols* from an **alphabet.**

  Think about words from the English language:

  - Alphabet: $\{a, A, b, B, c, C, \ldots, x, X, y, Y, z, Z\}$.
  - Example words: Hello, Coventry, and, or, a, ...

- Notation

Notation	Meaning	Example usage	
Σ	Alphabet: <u>finite</u> set of **symbols.**	$\Sigma = \{0, 1\}$	$\Sigma = \{\text{a}\}$
w or s	String made of symbols from Σ	01100	aaa
$\|w\|$	Length of the string w	$\|00\| = 2$	$\|a\| = 1$
ε	Empty string – has no symbols.	$\|\varepsilon\| = 0$	
xy	Concatenation of x and y	$x = 0, y = 10 \implies xy = 010$	

Concept of “**language**”

Think about words from the English language again:

- Alphabet: $\{a, A, b, B, c, C, \ldots, x, X, y, Y, z, Z\}$.
- However, not all strings over this alphabet are valid words.

  In English: *Hello* is valid, but *olleH* is not.
- Divide all possible instances into **yes-instances** and **no-instances**.
- $\rightarrow$ English is the **set** of “yes-instances over its alphabet.”
- $\rightarrow$ English is a **subset** of “all possible strings over its alphabet.”

Concept of "**language**"

In general:

- A (decision) **problem** is a **set of instances** and a required **property**.
- Each problem **instance** is represented by a **string** over an **alphabet** Σ.
- A **yes-instance** satisfies the property required by the problem.
- A **no-instance** does not satisfy the property required by the problem.

- The set of yes-instances defines a **language** associated to the **problem**.

- We say that the yes-instances *belong* to the language.
- No-instances (including invalid strings) *do not belong* to the language.

Language recognition

Decision problems can be encoded as problems of **language recognition.**

Problem: Is a given number **even**?

Instance: A number n (encoded in binary).
Question: Is n even? (i.e. is it divisible by 2?)

Example

- Given $n = 12_{10} = 1100_2$, the answer is **yes** because $12 = 2 \times 6$.
- Given $n = 13_{10} = 1101_2$, the answer is **no** because $13 = 2 \times 6 + 1$.

Here:

$$\begin{aligned} Numbers &= \{0, 1, 10, 11, 100, 101, 110, 111, 1000, \ldots\} \\ Even &= \{0, 10, 100, 110, 1000, \ldots\} \end{aligned}$$

and

$$Even \subset Numbers$$

Language recognition

Decision problems can be encoded as problems of **language recognition.**

Problem: Is a given number **even**?

Instance: A number n (encoded in binary).
Question: Is n even? (i.e. is it divisible by 2?)

- n can be represented as a string in binary using only two symbols: 0, 1.
- We can write a **decision procedure** to decide if this string belongs to the language of **yes** instances.

```
b ← least significant bit of n.
if b = 0 then
    return yes
else
    return no
end if
```

Terminology

- Languages are defined over an **alphabet** Σ.
- Σ^*: set of all possible strings over Σ, whose **length is finite.** ("*Sigma star*")

If $\Sigma = \{0, 1\}$ then

$$\Sigma^* = \{ \underbrace{\varepsilon}_{\text{Length 0}}, \underbrace{0, 1}_{\text{Length 1}}, \underbrace{00, 01, 10, 11}_{\text{Length 2}}, \underbrace{000, 001, 010, 011, 100, 101}_{\text{Length 3}}, \ldots\}$$

- A language can be regarded as "a subset of Σ^*".

Example

If $\Sigma = \{0, 1\}$ then the language of even numbers $Even \subset \Sigma^*$ is:

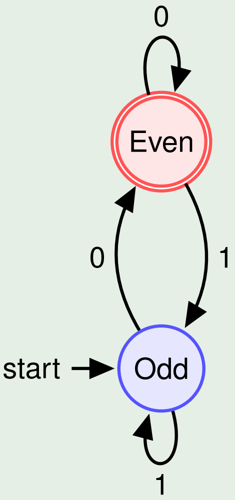
$$Even = \{0, 00, 10, 000, 010, 100, \ldots\}$$

Concept of “**Model of Computation**”

- We want to think more precisely about **problems** and **computation.**
- $\rightarrow$ categorise them by the **type of computation** which resolves them.
- $\rightarrow$ idea of **models** of computation:
  We introduce simple, theoretical machines and study their limits.
  - Far simpler than Von Neumann Machines, . . .
  - . . . but some have greater power than Von Neumann machines, . . .
  - . . . . . . but cannot be created in reality!
- Our first model is the **Deterministic Finite Automaton** (DFA) model.

The **Deterministic Finite Automaton** (DFA) model

Example (Is a given binary number even?)



Decimal	Binary
0	0
1	1
2	10
3	11
4	100
5	101
6	110
7	111
8	1000
9	1001
10	1010
11	1011
12	1100

Informal definition of DFAs

The **Deterministic Finite Automaton** (DFA) model

A **directed and labelled graph** which describes how a string of symbols from an alphabet will be processed.

- Each vertex is called a **state**.
- Each directed edge is called a **transition**.
  - The edges are labelled with symbols from the alphabet.
- Each state must have **exactly one** transition defined for **every** symbol.
- <u>One</u> state is designated as the **start state**.
- <u>Some</u> states are designated as **accept states**.
- A string is processed symbol by symbol, following the respective transitions:
  - At the end, if we land on an accept state then the string is accepted,
  - otherwise it is rejected.

Important rules for DFAs

- Each state must have **exactly one transition defined *for each symbol***.
- There must be **exactly one start state.**
- There may be **multiple accept states.**
- There may be more than one symbol defined on a single transition.

JFLAP simulation time!

Example

Let us build DFAs over the alphabet $\{0, 1\}$ to recognize strings that:

- begin with 0;
- end with 1;
- either begin **or** end with 1;
- begin with 1 **and** contain at least one 0.

Formal definition of DFAs

Formal definition of a DFA

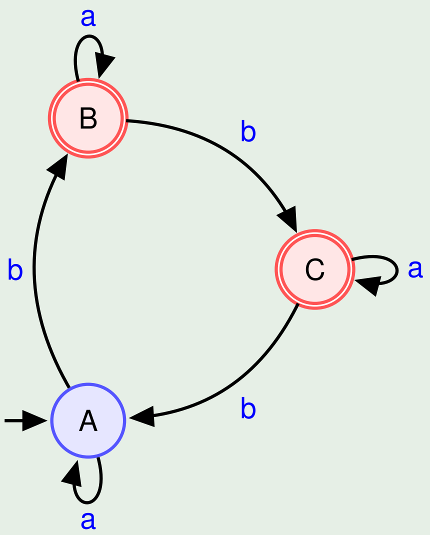
A *Deterministic Finite Automaton* (DFA) is defined by the 5-tuple $(Q, \Sigma, \delta, q_{\text{start}}, F)$ where:

- Q is a finite set called the **set of states.**
- Σ is a finite set called the **alphabet**.
- $\delta\colon Q \times \Sigma \to Q$ is a total function called the **transition function.**
- q_{start} is the unique **start state.** ($q_{\text{start}} \in Q$)
- F is **the set of accepting states.** ($F \subseteq Q$)

Recall:

- **Total function** means it is defined for "all its inputs."
- Σ, δ: Sigma, delta. (Greek letters)
- $\in, \subseteq$: "**element** of a set", "**subset** of a set, or equal". (Set notation)

Example (Formal specification of a DFA)



This DFA is defined by the 5-tuple $(Q, \Sigma, \delta, q_{start}, F)$ where

- $Q = \{A, B, C\}$
- $\Sigma = \{a, b\}$
- $\delta(state, symbol)$ is given by the table:

	a	b
$\rightarrow$ A	A	B
$*$ B	B	C
$*$ C	C	A

  $\rightarrow$ indicates the start state
  $*$ the accept state(s).

- $q_{start} = A$
- $F = \{B, C\}$

Notation: Functions/Maps

$\delta \colon Q \times \Sigma \to Q$ means that:

- the function δ takes a pair (q, s) as input where:
  - q is a *state* from Q
  - s is an *alphabet symbol* from Σ,
- and returns a state from Q as the result.

This is usually given as a table, e.g.

	a	b
$\rightarrow q_0$	q_0	q_1
$*q_1$	q_0	q_2
$\vdots$	$\vdots$	$\vdots$

We put $\rightarrow$ next to the start state, and $*$ next to the accept states.

This means that:

$$\begin{aligned} \delta(q_0, a) &= q_0 \\ \delta(q_0, b) &= q_1 \\ \vdots &= \vdots \end{aligned}$$

Recall: Power set – set of all subsets

2^Q is the **set of all subsets of** Q (called: the **power set of** Q)

Example

If $Q = \{A, B, C\}$ then

$$2^Q = \left\{ \underbrace{\emptyset}_{\text{Empty set}}, \underbrace{\{A\}, \{B\}, \{C\}}_{\text{One element each}}, \underbrace{\{A, B\}, \{A, C\}, \{B, C\}}_{\text{Two elements each}}, \underbrace{\{A, B, C\}}_{Q} \right\}.$$

It has 8 elements: $2^{\text{size of } Q} = 2^{\#Q} = 2^3 = 8$.

The **Nondeterministic Finite Automaton (NFA)** model

From the design point of view: NFAs are almost the same as DFAs.

DFA: every state has **one and only one outward transition** defined **for each symbol**.

NFA: every state has **zero or more transitions** defined **for each symbol**.

Formally:

DFA: $\delta\colon Q \times \Sigma \to Q$ is a **total** function, i.e.

1. δ is defined for *every* pair (q, s) from $Q \times \Sigma$
2. δ sends (q, s) to a **state** from Q. (exactly one state, no more, no less)

NFA: $\delta\colon Q \times \Sigma \to 2^Q$ is a **partial** function, i.e.

1. δ is *not necessarily* defined for every pair (q, s) from $Q \times \Sigma$.
2. δ sends (q, s) to a **subset of** Q. (many, one, or no states)

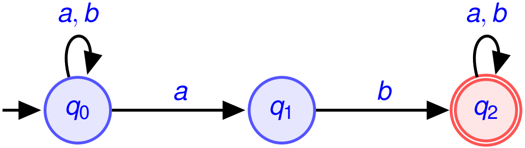
Formal definition of NFAs

Definition of an NFA

A *Nondeterministic Finite Automaton* (NFA) is defined by the 5-tuple $(Q, \Sigma, \delta, q_{\text{start}}, F)$ where

- Q is a finite set called the **set of states**
- Σ is a finite set called the **alphabet**
- $\delta : Q \times \Sigma \to 2^Q$ is a partial function called the **transition function**
- q_{start} is the unique **start state.** $(q_0 \in Q)$
- F is the **set of accepting states.** $(F \subseteq Q)$

NFA example

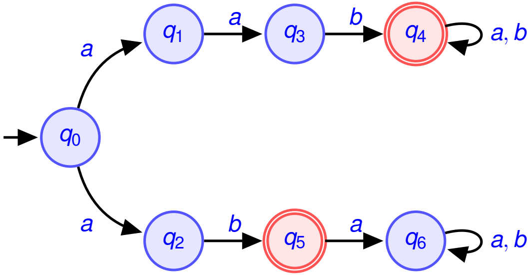


$$Q = \{q_0, q_1, q_2\}$$
$$\Sigma = \{a, b\}$$
$$q_{\text{start}} = q_0$$
$$F = \{q_2\}$$

δ :

	a	b
$\rightarrow q_0$	$\{q_0, q_1\}$	$\{q_0\}$
q_1	$\emptyset$	$\{q_2\}$
$*q_2$	$\{q_2\}$	$\{q_2\}$

NFA example



δ :

	a	b
$\rightarrow q_0$	$\{q_1, q_2\}$	$\emptyset$
q_1	$\{q_3\}$	$\emptyset$
q_2	$\emptyset$	$\{q_5\}$
q_3	$\emptyset$	$\{q_4\}$
$* q_4$	$\{q_4\}$	$\{q_4\}$
$* q_5$	$\{q_6\}$	$\emptyset$
q_6	$\{q_6\}$	$\{q_6\}$

$$
\begin{aligned}
Q &= \{q_0, q_1, q_2, q_3, q_4, q_5, q_6\} \\
\Sigma &= \{a, b\} \\
q_{\text{start}} &= q_0 \\
F &= \{q_4, q_5\}
\end{aligned}
$$

JFLAP simulation time!

Example

Let us build DFAs over the alphabet $\{0, 1\}$ to recognize strings that:

- begin with 0;
- end with 1;
- either begin **or** end with 1;
- begin with 1 **and** contain at least one 0.

Next week. . .

Surprise: NFAs recognize exactly the same languages as DFAs!