If there are any symbols or terminology you do not recognize then please let us know.

(1) Give the truth table for the following propositions

Expression	**Meaning**
$a \wedge b$	a and b
$a \vee b$	a or b
$a \oplus b$	a xor b
$\neg a$ (or $\bar{a}$)	not a
$a \implies b$	a implies b, or: if a then b
$a \iff b$	a and b are equivalent, or: "a if and only if b"

It is usual to apply these "bit-wise" to the bits of integers, e.g. $\texttt{0011} \oplus \texttt{0101} = \texttt{0110}$.

Solution

a	$\neg a$
0	1
1	0

a	b	$a \wedge b$	$a \vee b$	$a \oplus b$	$a \implies b$	$a \iff b$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

Think of:

- $\wedge$ as: *multiplication*.
- $\vee$ as: *addition*.
- $\oplus$ as: *difference* or *distance*.
- $\implies$ as: "*true* $\implies$ *false*" *is not allowed*.
- $\iff$ as: *equality*.

In particular, "$a \iff b$" is equivalent to "$a \Rightarrow b$ **and** $b \Rightarrow a$."
("$b \Rightarrow a$" can also be written as "$a \Leftarrow b$"). Written formally,

$$a \iff b \quad \equiv \quad (a \Rightarrow b) \wedge (b \Rightarrow a)$$

This can be shown using a truth table as follows:

a	b	$a \implies b$	$b \implies a$	$(a \implies b) \wedge (b \implies a)$	$a \iff b$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

We often use this latter fact to *prove* that two statements are equivalent. That is, if we want to prove that A and B are equivalent then we prove: $A \implies B$ and $B \implies A$.

(2) Recall that:

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of **natural numbers**
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of **integers**.

Consider the following set definitions

- $A = \{a \in \{1, 2, 3, 4\} \mid (a < 2) \vee (a > 3)\}$
- $B = \{a \in \mathbb{N} \mid a < 9\}$
- $C = \{a \in \mathbb{N} \mid a > 2 \wedge a < 7\}$
- $D = \{i \in \mathbb{Z} \mid i^2 \leq 9\}$

a) Give an explicit enumeration for each set, i.e. write down the elements in the form $\{x_1, x_2, \ldots\}$.

b) What is the cardinality of each set?

c) Which of these sets are subsets of at least one other set?

Solution

a)
- $A = \{1, 4\}$
- $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- $C = \{3, 4, 5, 6\}$
- $D = \{-3, -2, -1, 0, 1, 2, 3\}$

b)
- $\#A = 2$ (($\#A$ is also denoted by $|A|$))
- $\#B = 8$
- $\#C = 4$
- $\#D = 7$

c) $A \subset B$ and $C \subset B$.

(3) Write formal descriptions of the following sets.

a) The set containing all natural numbers that are less than 5.

b) The set containing all integers that are greater than 5.

c) The set containing the strings `aa` and `ba`.

d) The set containing the empty string.

e) The set containing nothing at all.

f) The set containing all the even integers.

Solution

a) $\{1, 2, 3, 4\} \quad = \quad \{n \in \mathbb{N} \mid n < 5\}$

b) $\{6, 7, 8, \dots\} \quad = \quad \{n \in \mathbb{N} \mid n > 5\} \quad = \quad \{n \in \mathbb{N} \mid n \geq 6\}$

c) $\{\texttt{aa}, \texttt{ba}\}$

d) $\{\varepsilon\}$

e) $\emptyset \quad = \quad \{\}$

f) $\{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\} \quad = \quad \{2k \mid k \in \mathbb{Z}\}$

The sets containing "…" are informal, and are only used to help with intuition.

(4) If the set A is $\{1, 3, 4\}$ and the set B is $\{3, 5\}$, write down:

Expression	**Meaning**
$A \cup B$	union of A and B
$A \cap B$	intersection of A and B
$A - B$	A minus B
$A \times B$	Cartesian product of A and B: set of all possible pairs (a, b) where $a \in A$ and $b \in B$
2^B (or $\mathcal{P}(B)$)	power set of B: set of all subsets of B

Solution

- $A \cup B = \{1, 3, 4, 5\}$
- $A \cap B = \{3\}$
- $A - B = \{1, 4\}$
- $A \times B = \{(1, 3), (1, 5), (3, 3), (3, 5), (4, 3), (4, 5)\}$
- $2^B = \{\emptyset, \{3\}, \{5\}, \{3, 5\}\}$

(5) Let X be the set $\{1, 2, 3, 4, 5\}$ and Y be the set $\{6, 7, 8, 9, 10\}$.

The *unary* function $f\colon X \to Y$ and the *binary* function $g\colon (X \times Y) \to Y$ are described in the following tables:

n	$f(n)$
1	6
2	7
3	6
4	7
5	6

g	6	7	8	9	10
1	10	10	10	10	10
2	7	8	9	10	6
3	7	7	8	8	9
4	9	8	7	6	10
5	6	6	6	6	6

- What are the *range* and *domain* of f?
- What are the *range* and *domain* of g?
- What is the value of $f(2)$?
- What is the value of $g(2, 10)$?
- What is the value of $g(4, f(4))$

Solution

$$f\colon \underbrace{X}_{\text{Domain}} \to \underbrace{Y}_{\text{Range}} \qquad g\colon \underbrace{X \times Y}_{\text{Domain}} \to \underbrace{Y}_{\text{Range}}$$

- Range of f: Y. Domain of f: X.
- Range of g: Y. Domain of g: $X \times Y$.
- $f(2) = 7$ (through table lookup).

n	$f(n)$
1	6
2	7
3	6
4	7
5	6

- $g(2, 10) = 6$ (through table lookup).

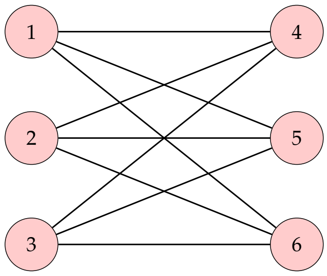
g	6	7	8	9	10
1	10	10	10	10	10
2	7	8	9	10	6
3	7	7	8	8	9
4	9	8	7	6	10
5	6	6	6	6	6

- $g(4, f(4)) = g(4, 7) = 8$.

n	$f(n)$
1	6
2	7
3	6
4	7
5	6

g	6	7	8	9	10
1	10	10	10	10	10
2	7	8	9	10	6
3	7	7	8	8	9
4	9	8	7	6	10
5	6	6	6	6	6

(6) Write a formal description of the following graph.



Solution

$G = (V, E)$ where $V = \{1, 2, 3, 4, 5, 6\}$ is the set of vertices, and the set of edges is

$$E = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$$

N.B. This graph is undirected, so technically the edges should be represented as sets rather than pairs (because the order is not important, e.g. the first edge should be $\{1, 4\}$ rather than $(1, 4)$) but we will tolerate this.

(7) Draw the (undirected) graph $G = (V, E)$, where

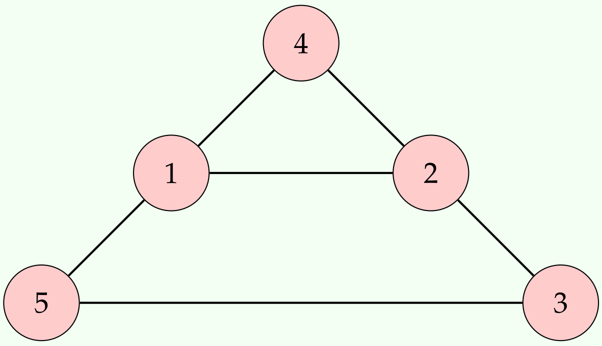
$$\begin{aligned} V &= \{1, 2, 3, 4, 5\} \\ E &= \{(1, 2), (1, 4), (2, 3), (2, 4), (3, 5), (1, 5)\} \end{aligned}$$

a) Is the graph connected?

b) What about the graph $G' = (V', E')$, where $V' = \{1, 2, 3, 4\}$ and $E' = \{(1, 3), (2, 4)\}$?

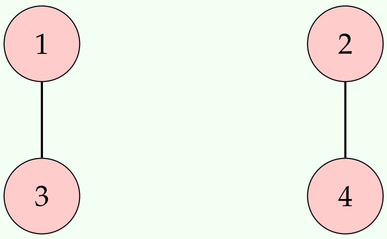
Solution

G:



G is connected.

G': (G' is pronounced "G prime")



G' is not connected.

(8) Draw the graph $G = (V, E)$, where $V = \{1, \ldots, 5\}$ and

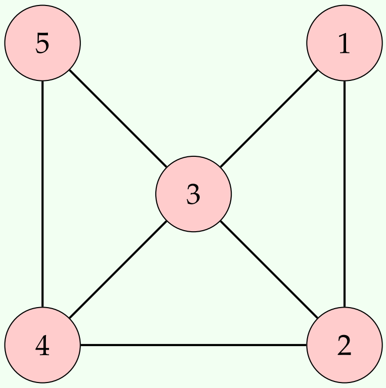
$$E = \{(a, b) \mid a, b \in V \wedge (a < b < a + 3)\}.$$

Solution

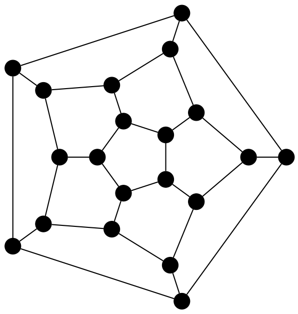
We need to find the pairs (a, b) that satisfy $a < b < a + 3$.
We can do this in table form:

a	b	Pairs (a, b)
1	2, 3	(1, 2), (1, 3)
2	3, 4	(2, 3), (2, 4)
3	4, 5	(3, 4), (3, 5)
4	5	(4, 5)
5		

e.g. when $a = 1$ we get $1 < b < 4$, so $b \in \{2, 3\}$, which gives us two pairs: $(1, 2)$ and $(1, 3)$.



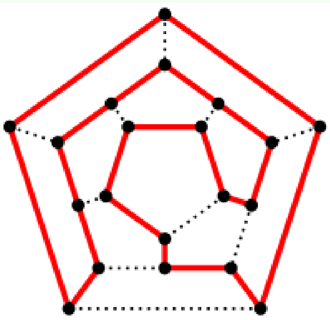
(9) The "Icosian Game" is a 19$^{\text{th}}$-century puzzle invented by the Irish mathematician Sir William Hamilton (1805–1865). The game was played on a wooden board with holes representing major world cities and grooves representing connections between them (see figure below).



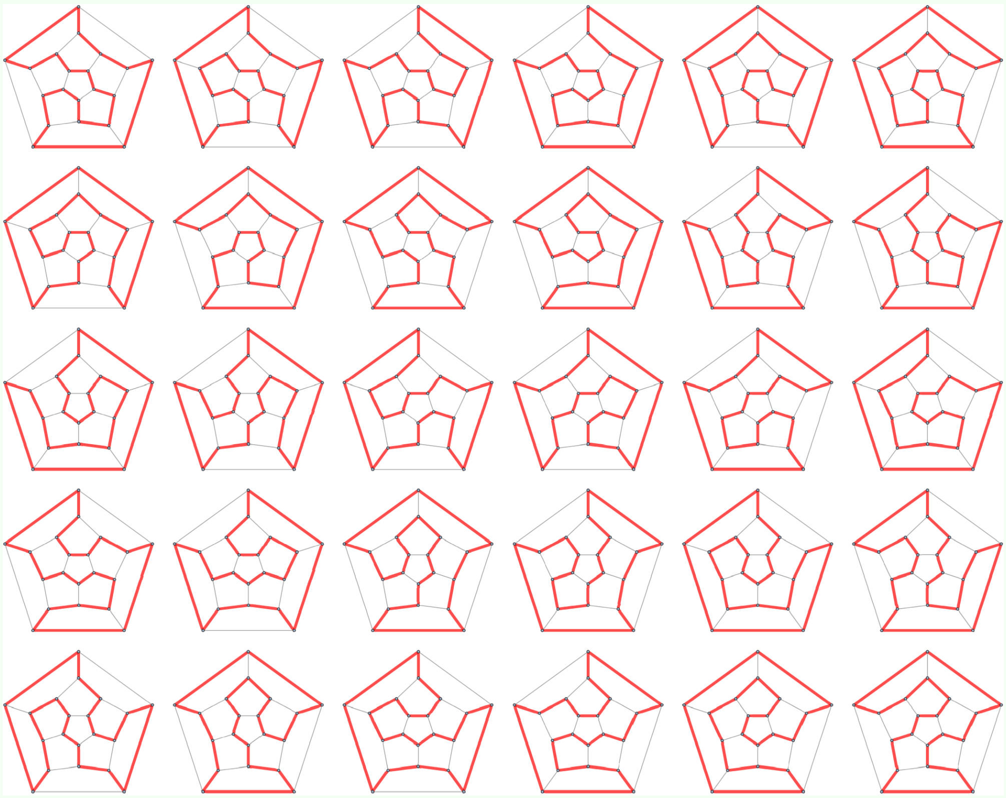
The object is to find a cycle that would pass through all the cities exactly once before returning to the starting point. Can you find such routes?

Solution

One possible solution is:



Here are the variations (see this article on MathWorld):



You may have noticed that these are "essentially" the same. This is called "symmetry."

Martin Gardner, a popular mathematics author, wrote:

> On a dodecahedron with unmarked vertices there are only two Hamiltonian circuits that are different in form, one a mirror image of the other. But if the corners are labeled, and we consider each route "different" if it passes through the 20 vertices in a different order, there are 30 separate circuits, not counting reverse runs of these same sequences.

A perspective projection of a dodecahedron with a Hamiltion cycle through its vertices (Source):