Introduction — Problems!

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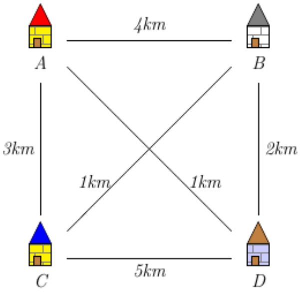
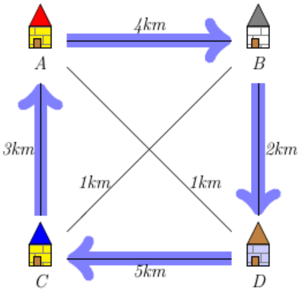
Lecture 1

Travelling Salesman Problem



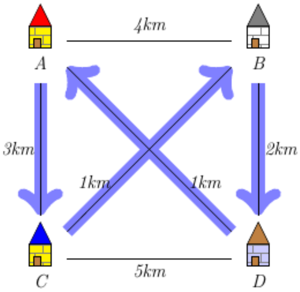
Shortest tour?

Travelling Salesman Problem



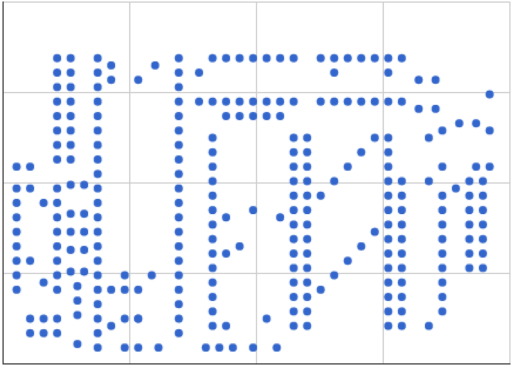
$$4 + 2 + 5 + 3 = 14$$

Travelling Salesman Problem

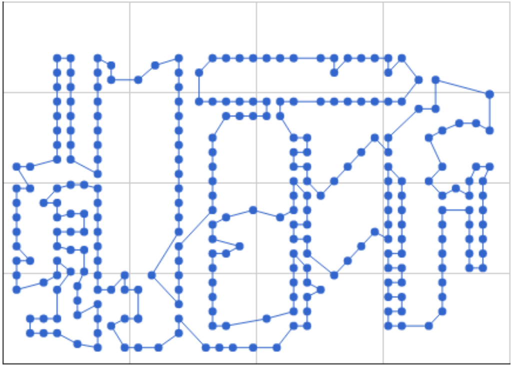


$$3 + 1 + 2 + 1 = 7$$

Travelling Salesman Problem



Travelling Salesman Problem



Travelling Salesman Problem

- One of the most famous problems in CS.

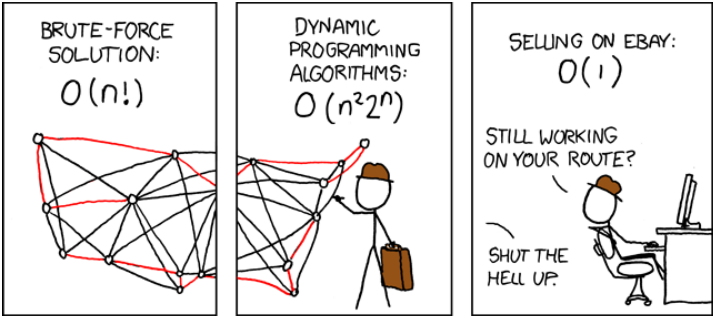
  *Given a **list of cities** and the **distances between each pair of cities**, what is the shortest possible route that visits each city and returns to the origin city?*

- “**NP-hard**” problem!

Travelling Salesman Problem – what is the issue?

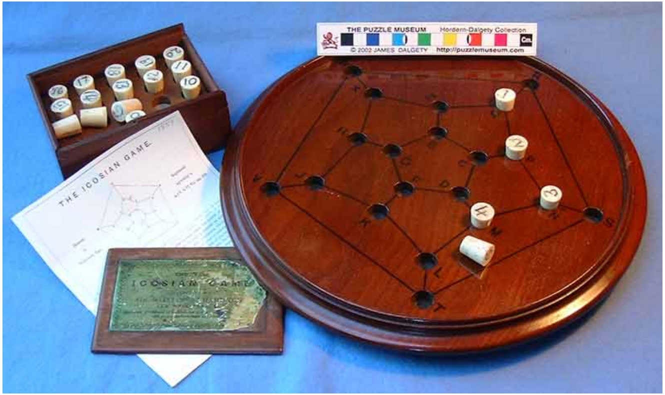
Number of cities n	Number of paths $(n-1)!/2$
3	1
4	3
5	12
6	60
7	360
8	$2,520$
9	$20,160$
10	$181,440$
15	$43,589,145,600$
20	6.082×10^{16}
71	5.989×10^{99}

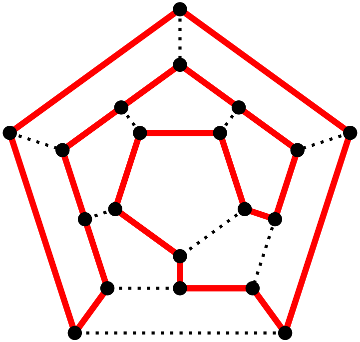
Travelling Salesman Problem – what is the issue?



Icosian Game – Hamiltonian Cycle Problem

by the Irish mathematician William Hamilton (1805–1865)





Problem (Hamiltonian Cycle)

Given a graph, decide if it contains a path that visits every vertex exactly once and terminates at the same starting vertex.

Subset-Sum Problem

Problem (Subset-Sum Problem)

Given a set $S = \{x_1, x_2, \ldots, x_n\}$ *of integers, and an integer* t *(called target) decide if there is a subset of* S *whose sum is equal to* t*.*

Example

Given the set $S = \{2, 3, 5, 7, 11, 13\}$, decide if there is a subset of S whose sum is 15.

Partition Problem

Problem (Partition Problem)

Given a set $S = \{x_1, x_2, \ldots, x_n\}$ *of numbers, decide if it can be partitioned into two sets such that they both have the same sums.*

Example

Given the set $S = \{2, 3, 5, 7, 11, 13\}$, is it possible to split it into 2 sets with equal sums?

Some more examples

Problem (Cliques)

Given a graph and an integer n, decide if it contains a clique with k vertices.

A clique in a graph is a set of vertices for which any two are connected.

Problem (A Diophantine quadratic equation)

Given three positive integers a, b, c, decide if the equation $ax^2 + by = c$ has a solution in positive integers.

Problem (Satisfiability)

Given a Boolean expression, decide if there is a way of assigning the values true and false to the variables so that the expression is true.

A needle in a haystack — a search problem

Problem:

Given any (finite) haystack H, decide whether H contains a needle.

Exhaustive Search

Search every location within the haystack, in some order, and terminate answering **yes** if a needle is found.
If the search is *completed* with no needle found then terminate answering **no**.

This problem is a **decision problem**: given some data (the haystack) decide if the data has a certain property (needle containment).

We may divide all possible instances of the problem into **yes-instances** and **no-instances** using our process.

Types of problems

- Decision
- Search
- Computation/Construction
- Counting
- Optimization
- ...

Important observation

As far as “Can these problems be solved at all using computation?” they can be reduced to **decision problems**.

Problems vs Problem Instances

1. What is $1 + 1$?
   $\rightarrow$ *instance* of the problem called **Addition Problem.**
   - not interested in just $1 + 1$, but $x + y$ in general.
2. What is the shortest route across the rail network from Coventry to London?
   $\rightarrow$ *instance* of the **Shortest Path Problem**,
3. What is the shortest tour around all the universities in the UK and back to your starting point (by car say)?
   $\rightarrow$ *instance* of the **Travelling Salesman Problem.**

Problem: Generalization of a problem *instance*.

Notation: Greek alphabet

α	alpha
β	beta
γ	gamma
δ	delta
ε	epsilon
σ	sigma
Σ	Sigma
Γ	Gamma

Notation: Numeric

$=$	equals
$\neq$	not equal
$<$	less than
$\leq$	less than or equal
$>$	greater than
$\geq$	greater than or equal
$n!$	Factorial of n: $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$

Notation: Logic

Expression	Meaning
$a \wedge b$	a and b
$a \vee b$	a or b
$a \oplus b$	a xor b
$\neg a$ (or $\bar{a}$)	not a
$a \implies b$	a implies b, or: if a then b
$a \iff b$	a and b are equivalent, or: "a if and only if b"

Notation: Sets

$\{x_1, \ldots, x_n\}$	Finite set consisting of the elements x_1 until x_n
$\emptyset$	Empty set, i.e. $\{\}$
$x \in S$	"in", x is a member of the set S
$x \notin S$	"not in", x is not a member of the set S
$A \cup B$	Union of the two sets A and B
$A \cap B$	Intersection of the two sets A and B
$A - B$	Difference of the two sets A and B
$A \times B$	Cartesian product of the two sets A and B
$A \subset B$	A is a subset of B
$\|A\|$ or $\#A$	Cardinality of the set A, i.e. count of its elements
2^A	Power set of A, i.e. set of all subsets of A
$\mathbb{N}$	Natural numbers: $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$	Integers: $\{0, 1, -1, 2, -2, 3, -3, \ldots\}$
S'	A set called "S prime" (a way of making new names)
S'' and S'''	Sets called "S double prime" and "S triple prime"

Notation: Sets and logic notation

$\{pattern \mid condition\}$ Set of items matching *pattern* and satisfying *condition*.
The $\mid$ symbol is read "such that"

$$
\begin{aligned}
A \cup B &= \{x \mid x \in A \lor x \in B\} \\
A \cap B &= \{x \mid x \in A \land x \in B\} \\
A - B &= \{x \mid x \in A \land x \notin B\} \\
A \times B &= \{(a, b) \mid a \in A \land b \in B\}
\end{aligned}
$$

Notation: Functions

$$\begin{aligned} f\colon A &\to B \\ x &\mapsto y \end{aligned}$$

- f is a function that takes input x from the set A and returns an element y from B.
  - We say: "f **maps** x to y" and write $f(x) = y$.
- A is the **domain** of f, the set of possible inputs.
- B is the **range** of f, the set of possible outputs.

Similarly

$$\begin{aligned} f\colon X \times Y &\to R \\ (x, y) &\mapsto r \end{aligned}$$

- f is a function that takes as input a pair (x, y) from the set $X \times Y$ and returns an element r from R.

Notation: Strings

Notation	Meaning	Example
Σ	Finite set of symbols	$\Sigma = \{0, 1\}$
w	String made of symbols from Σ	010
$\lvert w\rvert$	Length of the string w	$\lvert 010\rvert = 3$

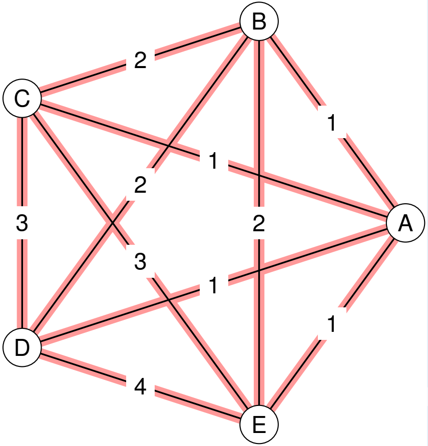
Notation: Graphs

$G = (V, E)$, where

- V: the set of **vertices.**
- E: the set of **edges.**

Here:

- $V = \{A, B, C, D, E\}$
- $E = \{(A, B), (A, C), (A, D), (A, E), (B, C), (B, D), (B, E), (C, D), (C, E), (D, E)\}$



Notation: Graphs

Graph can be:

- **directed** or **undirected**.
- **weighted** or **unweighted**.
- **labelled** or **unlabelled**.
- etc.

Properties:

- Is the graph **connected**?
- Does it contain **cycles**?
- etc.

Algorithms:

- Traversal, e.g. BFS, DFS.
- Shortest path.
- etc.

Next few weeks. . .

What is a “computer”? What is “computation”?