Recall the problems:

$$\text{SAT} = \{\langle\phi\rangle \mid \phi \text{ has at least } \textit{one} \text{ satisfying assignments}\}$$

and

$$\text{DOUBLE-SAT} = \{\langle\phi\rangle \mid \phi \text{ has at least } \textit{two} \text{ satisfying assignments}\}$$

(1) For the following Boolean formulae, count how many satisfying assignments they have, then decide if they are in SAT and DOUBLE-SAT?

- $\phi_1 = x \wedge (y \vee \bar{x}) \wedge (z \vee \bar{y})$
- $\phi_2 = (x \vee y) \wedge \big((\overline{x \vee z}) \vee (\bar{z} \wedge \bar{y})\big)$
- $\phi_3 = (\bar{x} \vee y \vee z) \wedge (x \vee \bar{y} \vee z) \wedge (x \vee y \vee \bar{z})$

Basic

(2) Define

$$\text{TRIPLE-SAT} = \{\langle \phi \rangle \mid \phi \text{ has at least 3 satisfying assignments}\}$$

Complete the proof below to show that TRIPLE-SAT is **NP-complete**.

TRIPLE-SAT is NP-complete

We need to show that TRIPLE-SAT $\in$ **NP** [ ] SAT $\leq_P$ TRIPLE-SAT.

1/2 TRIPLE-SAT $\in$ **NP**:
On input $\langle \phi(x_1, \ldots, x_n) \rangle$, non-deterministically guess [ ] different assignments for the Boolean variables $x_1, \ldots, x_n$, and verify whether they all [ ] ϕ.
The verification step only costs $O($ [ ] $)$.

2/2 SAT $\leq_P$ TRIPLE-SAT:
On input $\langle \phi(x_1, \ldots, x_n) \rangle$, introduce 2 new Boolean variables a and b, and output the formula:

$$\Psi(x_1, \ldots, x_n, a, b) = \phi(x_1, \ldots, x_n) \wedge (a \vee b).$$

- (1/2) If $\langle \phi \rangle \in$ SAT then this means that ϕ has at least [ ] satisfying assignment, and therefore $\phi \wedge (a \vee b)$ has at least [ ] satisfying assignments because the added clause $(a \vee b)$ can be satisfied in 3 ways:

$$(a, b) \in \{(1,0), (0,1), \boxed{}\}.$$

So $\langle \Psi \rangle = \langle \phi \wedge (a \vee b) \rangle \in$ TRIPLE-SAT.

- (2/2) If $\langle \phi \rangle \notin$ SAT then $\phi \wedge (a \vee b)$ [ ] have a satisfying assignment because

$$0 \wedge 0 = 0 \qquad \text{and} \qquad 0 \wedge \boxed{} = 0.$$

So $\langle \Psi \rangle = \langle \phi \wedge (a \vee b) \rangle \notin$ TRIPLE-SAT.

Therefore, SAT $\leq_P$ TRIPLE-SAT, and hence TRIPLE-SAT is **NP-complete**.

(3) Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers, and define

$$\mathbb{N}_t = \{n \in \mathbb{N} \mid n < t\} \qquad \text{(i.e. } \mathbb{N}_t = \{1, 2, \ldots, t-1\}\text{)}$$

Consider the following restricted version of the **Subset-Sum Problem** (SSP)

$$\text{SSP} = \{\langle \mathcal{S}, t\rangle \mid \mathcal{S} \subsetneq \mathbb{N}_t \text{ is a finite set, and } t \in \mathbb{N}, \text{ such that there is a subset of } \mathcal{S} \text{ whose sum is } t\}$$

Here the notation "$A \subsetneq B$" means that $A \subset B$ but $A \neq B$.

SSP is **NP-complete**.

Define:

$$\text{DOUBLE-SSP} = \{\langle \mathcal{S}, t\rangle \mid \mathcal{S} \subset \mathbb{N}_t \text{ is a finite set, and } t \in \mathbb{N}, \text{ such that there are two distinct subsets of } \mathcal{S} \text{ which both sum to } t\}$$

Complete the proof below to show that DOUBLE-SSP $\in$ **NP-complete**.

DOUBLE-SSP is NP-complete

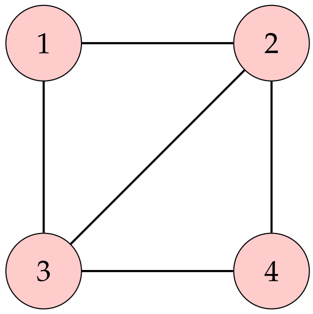
1/2 DOUBLE-SSP $\in$ **NP** because we can [ ] if 2 given subsets $\mathcal{T}_1, \mathcal{T}_2$ of $\mathcal{S}$ sum to t and check that $\mathcal{T}_1 \neq \mathcal{T}_2$ in time $O($ [ ] $)$, which is polynomial time.

2/2 SSP $\leq_P$ DOUBLE-SSP.
Since $\mathcal{S} \subsetneq \mathbb{N}_t$ then $\mathcal{S}$ [ ] at least an element m from $\{1, 2, 3, \ldots, t-1\}$. Select such an m and build a DOUBLE-SSP instance $\langle \mathcal{S}', t\rangle$ where $\mathcal{S}' = \mathcal{S} \cup \{m, t-m\}$.

- (1/2) Showing that: $\langle \mathcal{S}, t\rangle \in \text{SSP} \implies \langle \mathcal{S}', t\rangle$ [ ] DOUBLE-SSP.
Since $\langle \mathcal{S}, t\rangle \in$ SSP then there is a subset $\mathcal{T} \subset \mathcal{S}$ which [ ] to t.
A second solution is given by [ ].

- (2/2) Showing that: $\langle \mathcal{S}', t\rangle \in \text{DOUBLE-SSP} \implies \langle \mathcal{S}, t\rangle \in$ [ ].
If $\langle \mathcal{S}', t\rangle \in$ DOUBLE-SSP then there are two distinct subsets $\mathcal{T}$ and $\mathcal{U}$ of $\mathcal{S}'$ that both sum to t.
$\mathcal{T}$ and $\mathcal{U}$ cannot both be $\{m, t-m\}$ because they must be [ ]. So one of them must be a [ ] of $\mathcal{S}$.
So, $\langle \mathcal{S}, t\rangle \in$ SSP.

(4) Consider the following graph:



and recall that

$$\text{CLIQUE} = \{\langle G, k\rangle \mid \text{The graph } G \text{ has a } k\text{-clique}\}$$

- What is the largest k for which this graph satisfies CLIQUE?
- In general, how many edges does a k-clique have (as a function of k)?

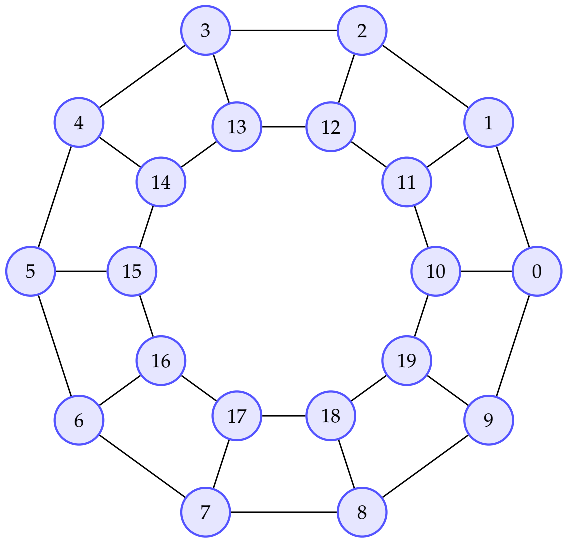
(5) A ***vertex cover*** of an undirected graph G is a subset of the vertices where every edge of G touches one of those vertices.

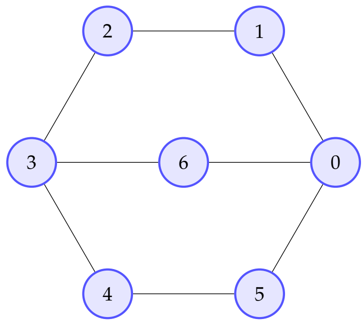
The VERTEX-COVER problem asks whether a graph contains a vertex cover of a specified size:

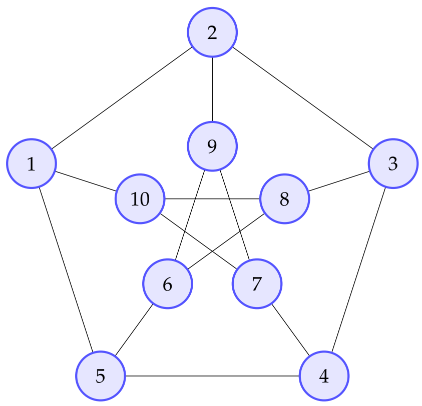
$$\text{VERTEX-COVER} = \{\langle G, k\rangle \mid G \text{ is an undirected graph that has a } k\text{-vertex cover}\}.$$

What is the smallest k for which the graph from the previous question satisfies VERTEX-COVER?

(6) Do the following graphs have Hamiltonian circuits?







(7) Let $x_1, x_2, \dots, x_n$ be Boolean variables, and let ϕ be a Boolean formula written in 3-cnf (Conjunctive Normal Form, like ϕ_3 in the first excercise) given by

$$\phi = c_1 \wedge c_2 \wedge \cdots \wedge c_\ell,$$

where each clause c_m has the form $\alpha \vee \beta \vee \gamma$, where each of α, β, γ is a literal: a variable x_i or its negation $\bar{x}_i$.

The 3SAT problem is **NP-complete**, and asks if a given 3-cnf formula is satifiable.

We showed in the lecture that the Subset-Sum Problem (SSP) is in **P**. Now, show that SSP is **NP-complete** by reducing 3SAT to it, i.e. show that

$$\text{3SAT} \leq_P \text{SSP}$$

You may find it easier if you study the discussion at https://saravananthirumuruganathan.wordpress.com/2011/02/07/detailed-discussion-on-np-completeness-of-subset-sum/ then the proof given in the textbook.

(8) Prove that CLIQUE is **NP-complete** by reducing the VERTEX-COVER problem to CLIQUE. (VERTEX-COVER is **NP-complete**)

You can use the following reduction from VERTEX-COVER to CLIQUE:

> Suppose we are given an instance $\langle G, k \rangle$ of VERTEX-COVER. Construct the instance $\langle G', n-k \rangle$ of CLIQUE, where n is the total number of nodes of G, and G' is G with the set of edges complemented (i.e. G' has an edge if and only if G does not have that edge).

Intermediate

(9) Given a graph G with an even number of vertices n, does G have an $n/2$-clique?

Hint: Reduce CLIQUE to HALF-CLIQUE. You need to figure out how to add vertices to adjust the size of the largest clique depending on whether $k = n/2$ or $k > n/2$ or $k < n/2$ (e.g., if $k = n/2$, just produce G).

(10) Show that the class **P** is closed under:

- Union.
- Concatenation.
- Complementation.

Intermediate